Further Risps

Rich tasks for the
Further Mathematics
A Level classroom

Jonny Griffiths

Further Risps

Forty investigative tasks
for the Further Mathematics classroom

www.further-risps.co.uk

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Please visit the Further Risps Facebook Page
to comment or feed back on the tasks.

Some people believe in imaginary friends. I believe in imaginary numbers.
R.M.ArceJaeger

If there is a God, he's a great mathematician.
Paul Dirac (1902-1984)

Always try the problem that matters most to you.
Andrew Wiles (1953-)

I am interested in mathematics only as a creative art.
G.H. Hardy (1877-1947)

It's the story that matters, and not just the ending.
Paul Lockhart

For Bernard

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Contents

Welcome

Hello, and welcome to this book of risps for Further Maths. The word 'risp' started out as an acronym for 'rich starting point', and it has been popularised by my site at **www.risps.co.uk**, a venture which began life back in 2005. My risps arose from a desire to make learning maths at A level more exploratory and open-ended without ignoring syllabus requirements. I was certainly not the first person to try to write open mathematics tasks (maybe even Plato is a contender); these have been authored by many people, both before and after my attempts, but maybe there is room for one more collection, this time for those studying Further Maths A Level, an area where open tasks are perhaps less common. Some might say it's harder to write such activities for the FM setting; I'm not sure that's true. Wherever you teach maths, at primary, secondary, sixteen plus or university level, creating investigatory materials like this is, I hope, always possible.

Please take the kernel of each risp idea, make it your own, and customise it for your own classroom situation. Only you know your learners and the context in which you teach them. Be as free as you like in your classroom with what's here.These tasks are designed to help with an active approach to learning, and encourage discussion that leads on to deeper understanding. I've tried to make each risp a gentle example of the kind of research work that university pure mathematicians might do. The level is, of course, different, but the process of playing with examples, edging towards some kind of theorem or conjecture, and working from there towards a proof of sorts, one that may need to be refined time and time again; for me, that's closer to what might be called 'real maths' than pages of repetitive exercises.

In the UK it's possible to study two Maths A Levels, Maths and Further Maths (the really keen student may wish to self-study for a third). A typical learner on such a course will be between 16 and 18 years of age, and is likely to be heading on to a mathematics-related degree. This book is intended firstly for those entrusted with teaching such students, although it may be useful to these same students too, if they choose to work independently. If that describes you, then good luck with these tasks; it could be that you'll be considering an EPQ (Extended Project Qualification), in which case there may be ideas here that you can nurture into a full submission. Some UK students don't take Further Maths for some reason, but still go on to study

Maths at University. This means that some of the tasks here may be suitable for the first year of a university course. My original risps were aimed at A Level maths teachers, but they also proved useful to those training teachers at university, which was a welcome surprise. I hope too that teachers from further afield might tune in to these activities, but I'm aware that every country has its own way to structure mathematics learning. Like any author, I'm scattering these pages to the winds, and I'll be delighted wherever they find good earth.

How best do people learn mathematics? It's a deep question, one I've lived with for many years as I've tried to teach my students, sometimes successfully and sometimes not, and as I've tried to learn the subject myself, with equally mixed results. Condensing all of those years into a key sentence to sum up the philosophy underlying these pages is not easy; I'm probably foolish to try, but here goes.

My Key Sentence

If you feel curiosity rising as you take on a mathematics question, if this question becomes truly yours, so that you long for a resolution, if you feel free to discuss what you find as you investigate with your colleagues in a spirit of mutual support, if you yearn for new theory that might allow you to see more deeply into this situation, then you are learning maths, as you play, and not in a rote way, but truly, in a way that could stay with you forever.

You will find here forty activities that are written with this underpinning philosophy in mind. These pages fall into the three-part pattern

1. Further Risps, 2. Further Risps; Reflections, 3. Further Risp Notes.

Each risp could be copied onto A3 or A4 for your students to work on in groups (some tasks may need editing for your syllabus); the margins here will hopefully be generous enough to accommodate plenty of working. At the end of the book are Notes (in the main possible answers) that offer outcomes from taking on the tasks. In the middle of the book there are several short reflections on the background to these activities and on how they might be used. There's a support website at **www.further-risps.co.uk**, and on the Further Risps Facebook page or via email you have a chance to leave feedback, which will be gratefully received (one becomes typo-blind after reading a book for the twentieth time). I'm hoping the second edition of this book will be more accurate than the first.

You may say my key sentence above is impossibly utopian. Some might claim the number of A Level students who are going to feel like this about their maths is bound to be limited. Perhaps, but would it not be good to add to their number? I'm not offering these activities as any kind of magic bullet. I don't think all teacher-talk is awful, and I don't see all teaching based around investigations as brilliant, especially when the tasks are not carefully chosen. But I do see my key sentence as describing what the core activity should be in a maths classroom, the centre around which everything should be built. After all, why did we become maths teachers in the first place? There are many pressures on us to lose the faith, especially when a utilitarian government comes along that doesn't understand, but we must stand firm. For many of us, learning maths has been the most thrilling and formative intellectual experience of our lives, and that's what we want to pass on. I hope you find at least some tasks that you like here, and that you enjoy using this book as much as I enjoyed writing it.

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Further Risps by Topic

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Technology requirements

Some of these tasks are simply pen and paper, others require technology.

Graphing packages

You could use a calculator (polar and Cartesian required) or a computer.

The most popular graphing packages available currently include **Desmos** (free), **Geogebra** (free) and **Autograph** (also free).

Whatever you use, you will need the ability to control the value of named constants in your equations dynamically, by using a slider or some other technique.

Computer algebra systems

Using CAS is helpful for certain tasks, where the alternative is plenty of drudgery.

I use the package **Derive**; I'm a big fan, but it's not as readily available to purchase as it used to be. You may be able to find a download somewhere. Texas Instruments have built Derive into their more advanced TI-Nspire machines, and I've bought this version of Derive from them before now as a stand-alone program. A faithful band of supporters remain in the shape of the Derive Users Group (DUG).

Geogebra has a basic computer algebra package built in, but it is at an early stage of development.

Note; calculators with CAS built in cannot currently be taken into most UK exams.

Programming

This crops up in a couple of places, and is an extremely helpful thing for A Level mathematics students to learn.

I use **VBA** as built into Excel spreadsheets. There is a mini-chapter on how to embark on this on page 103.

Spreadsheets

Excel is used elsewhere simply as a helpful spreadsheet package.

Dynamic geometry

These tasks don't involve much of this, but you may be able to extend them in ways that do. **Geogebra** is the program of choice here.

Online Engines

The free version of **Wolfram Alpha** can be handy for checking calculations; see **https://www.wolframalpha.com**

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Further Risp 1: The Twizzle

Level 1

Given a $2{\times}2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define the ***twizzle*** of $A = A^Z$ to be $\begin{pmatrix} b & d \\ a & c \end{pmatrix}$.

So the matrix A^Z is the matrix A with all its elements picked up and rotated anti-clockwise about the centre of the matrix through $90°$.

Choose and write down a non-singular 2×2 matrix A, and then write down A^Z.

What's the determinant of A? Of A^Z? How are they connected? Does this always work?

Level 2

Suppose you define the twizzle of a 3×3 matrix A in a similar way.

Write down a non-singular 3×3 matrix A, together with A^Z.

Can you find $\det(A)$ and $\det(A^Z)$?

You could use a calculator to check your calculations.

How are $\det(A)$ and $\det(A^Z)$ connected here?

Can you prove any conjectures you make?

Level 3

Key trig facts; true for all θ in radians.

$$\sin\theta = \cos\left(\tfrac{\pi}{2}-\theta\right), \cos\theta = \sin\left(\tfrac{\pi}{2}-\theta\right),$$
$$\sin(\pi-\theta) = \sin\theta, \cos(\pi-\theta) = -\cos\theta,$$
$$\sin(-\theta) = -\sin\theta, \cos(-\theta) = \cos\theta.$$

Write down the 2×2 matrices for a rotation through θ about the origin, and for reflection in the line $y = x\tan\theta$.

Suppose you're given a 2×2 rotation matrix R; can you say anything about what kind of matrix R^Z will be? How about R^{ZZ}?

Suppose now you're given a 2×2 reflection matrix F; can you say anything about what kind of matrix F^Z will be? How about F^{ZZ}?

Level 4

If A^T stands for 'the transpose of A', then the identity $(AB)^T = B^T A^T$ is true for all square matrices A and B of the same size (can you prove this?)

Can you show that $(AB)^{-1} = B^{-1}A^{-1}$ holds for all square matrices A and B of the same size?

In general, for all 2×2 matrices A and B, does $(AB)^Z = B^Z A^Z$?

If for all 2×2 matrices A you know that $(AB)^Z = B^Z A^Z$, what can you say about the 2×2 matrix B? If for all 2×2 matrices B you know that $(AB)^Z = B^Z A^Z$, what can you say about the 2×2 matrix A?

If for all 2×2 matrices A you know that $(AB)^Z = B^Z A^Z$ and $(BA)^Z = A^Z B^Z$, what can you say about the 2×2 matrix B?

The Notes for this risp are on page 110; Home Page

Further Risp 2: The Spiral-Line Area

Level 1

Using a graphing program, draw the polar curve $r = n\theta$, where n is a positive real number. What happens as n increases and decreases?

Now add to the graph the line $\theta = 1 + \dfrac{1}{n}$. What happens as n increases and decreases now?

Level 2

Can you find the area A enclosed by the curve above and the line above?

Does A have a stationary point as n varies? If so, for what value of n does this happen, and what is the nature of the point?

Can you generalise this result to the line $\theta = a + \dfrac{b}{n}$, where $a, b > 0$?

Level 3

Are there any other decreasing functions of n, say $f(n)$, where working with the boundary line $\theta = f(n)$ gives A a stationary point for some value of n?

What happens if you choose $\theta = e^{-n}$?

Level 4

Suppose the boundary line is $\theta = f(n)$, where f is such that A remains constant as n increases. Can you find $f(n)$?

The Notes for this risp are on page 113; Home Page

Further Risp 3: Can $a + bi = ae^{bi}$?

Level 1

If a and b are real numbers with $0 < a, 0 < b < \pi$, is it ever possible for

$$a + bi = ae^{bi}?$$

Level 2

If $a + bi = 2ae^{bi}$, with $0 < a, 0 < b < \pi$, what can you say about a and b now?

What if you try $a + bi = \dfrac{2}{\sqrt{3}}ae^{bi}$?

Level 3

Suppose now $a + bi = kae^{bi}$, where k is a real number, $k > 1$.

Can you always find a solution in terms of k for a and b here, for all $k > 1$?

If you can, what's the solution? If you can't, can you prove that you can't find one?

Level 4

If $a + bi = kae^{bi}$ as above, what is the locus of $a + bi$ in the Argand diagram as k varies?

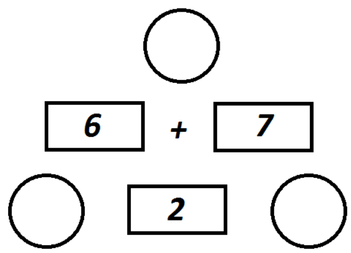
Can you use a computer graphing package or calculator to draw this curve?

The Notes for this risp are on page 116; Home Page

Further Risp 4: Series Arithmagon

Level 1

What's an arithmagon?



This is an addition arithmagon. Can you put numbers into the circles so that the two numbers either side of a rectangle add to the number it contains each time?

Will a solution exist whatever numbers you start with in the rectangles?

If so, will the solution be unique?

Level 2

It is often possible to expand a function $f(x)$ into an infinite sum of multiples of non-negative powers of x.

Sometimes this expansion holds for only a limited range of x-values.

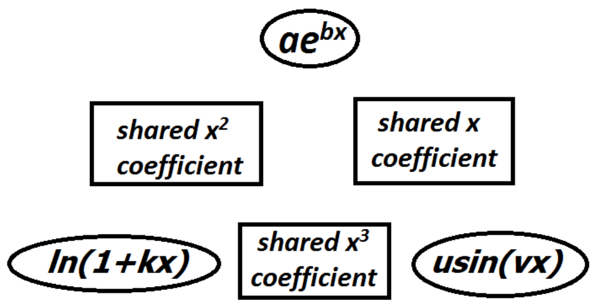
This infinite sum is called the Maclaurin expansion of $f(x)$, and is given by

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$

What is the Maclaurin expansion for e^x? For $\ln(1+x)$? For $\sin x$? For $\cos x$?

Can you research for which values of x these expansions are valid?

Level 3



The rectangles above refer to the coefficients in the Maclaurin functions of the functions in the ellipses.

Can you find distinct non-zero integers a, b, k, u, v so that the arithmagon above holds?

In what interval must x lie for your solution to be valid?

Level 4

Can you find an infinite family of integers a, b, k, u, v that work for this arithmagon?

Such a family could be of the form

$$(a, b, k, u, v) = (f_a(r, s), f_b(r, s), f_k(r, s), f_u(r, s), f_v(r, s)),$$

where each f_i is a polynomial function, and r and s are any integers (you would call r and s *parameters* here).

The Notes for this risp are on page 119; Home Page

Further Risp 5: Inverse Trig Triangles

Level 1

The function $\arcsin x$ (often on calculators as $\sin^{-1} x$) means 'the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sin is x' (this is using radians). The domain for arcsin is $-1 \leq x \leq 1$.

Find the value of $\arcsin x + \arccos x$ for a few values of x using a calculator. Can you explain this?

Level 2

Is it possible to find a value of x so that $\arcsin x, \arccos x$ and $\arctan x$ are the three angles of a triangle?

How about $2 \arcsin x, 2 \arccos x$ and $2 \arctan x$ for the three angles of a triangle?

How about $1.5 \arcsin x, 1.5 \arccos x$ and $1.5 \arctan x$ for the three angles of a triangle?

If k is such that $k \arcsin x, k \arccos x$ and $k \arctan x$ form the three angles of a triangle, what's x here, and what values can k take?

Level 3

Show that $\arctan a + \arctan b = \arctan \dfrac{a+b}{1-ab}$.

Find a similar expression for $\arctan a + \arctan b + \arctan c$.

If $\arctan x, \arctan 2x$ and $\arctan 3x$ are the angles of a triangle, what's x now?

Level 4

You will need a computer graphing package for this part.

Can $\arcsin x, \arcsin 2x$ and $\arcsin 3x$ be the angles of a triangle?

Can you find the coordinates of the endpoint of

$$y = \arcsin x + \arcsin 2x + \arcsin 3x$$

that lies in the first quadrant?

Can $\arccos x, \arccos 2x$ and $\arccos 3x$ be the angles of a triangle?

Can $2\arcsin x, 2\arcsin 2x$ and $2\arcsin 3x$ be the angles of a triangle?

What do you notice with these last two results?

Can you form a conjecture? Can you prove this?

The Notes for this risp are on page 122; Home Page

Further Risp 6: FP1 Odd One Out

Below you are given several collections, each of three things. Your task is to find a good reason why each of the things could be regarded as different from the other two.

Given the numbers 2, 3 and 9, for example, then 2 is the only even number, 9 is the only square, and 3 is the only triangle number.

Triplet 1

$$4x^3-20x^2+29x-28=0, 3x^3-17x^2+33x-21=0, 2x^3-10x^2+22x-25=0.$$

Triplet 2

$$\sum_1^n (5r+27), \sum_1^n (5r^2+2), \sum_1^n (3r^3+1).$$

Triplet 3

You are given three graphs:

$$x-y=3, x^2-y^2=3, x^3-y^3=3.$$

Try to sketch the graphs before using a graph plotter to confirm what they look like.

Triplet 4

$$y=\frac{3x^2+22x+16}{x^2+6x+8}, y=\frac{3x^2+18x+36}{x^2+8x+15}, y=\frac{4x^2+20x+10}{x^2+6x+5}.$$

Triplet 5

$$16e^{i\pi/3}, 2+2\sqrt{3}i, -8+8\sqrt{3}i.$$

Triplet 6

$$e^{2x}\ln(1+5x), e^{5x}, e^{2x}\cos(-3x).$$

Triplet 7
Remember! Working in degrees, for n an integer,

$$\begin{aligned}
&\cos a = \cos b \Rightarrow a = b + 360n \quad \text{or} \quad -b + 360n\\
&\sin a = \sin b \Rightarrow a = b + 360n \quad \text{or} \quad 180 - b + 360n\\
&\tan a = \tan b \Rightarrow a = b + 180n
\end{aligned}$$

For solutions in the range 5° to 20°;

$$\cos(660^\circ - 40x) = 0.5, \sin(18x - 135^\circ) = \frac{1}{\sqrt{2}}, \tan(22.5x - 195^\circ) = \frac{1}{\sqrt{3}}.$$

Triplet 8

Finding the approximate value of a root by

1. interval bisection
2. linear interpolation
3. the Newton-Raphson method.

Triplet 9

You wish to use a set of data for x and y to find the values of a and b by plotting a graph.You believe that the law in operation connecting x and y is of the form

$$1.y = ab^x, 2.y = ax^b, 3.y = xa^b.$$

Triplet 10

$$\begin{pmatrix}\sqrt{2} & -\sqrt{2}\\ \sqrt{2} & \sqrt{2}\end{pmatrix}, \begin{pmatrix}3 & -\sqrt{3}\\ \sqrt{3} & 3\end{pmatrix}, \begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}.$$

The Notes for this risp are on page 126; Home Page

Further Risp 7: The Which-Clues Matrix

Level 1

What do we mean by 'the trace $\mathrm{tr}(X)$ of a square matrix X'?

What do we mean by 'the characteristic equation of a square matrix X'?

Level 2

Can you show that the characteristic equation of a 3×3 matrix X is for some k

$$t^3 - \mathrm{tr}(X)t^2 + kt - \det(X) = 0$$

What would the corresponding equation be for a 2×2 matrix?

Level 3

Here are seven clues that may or may not be true for a 3×3 matrix X with real eigenvalues.

Clue A: one eigenvalue of X is 8.
Clue B: one eigenvalue of X is -5.
Clue C: The determinant of X is -160.
Clue D: The trace of X is 0.
Clue E: $X = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, where $a < b < c$.
Clue F: The eigenvectors of X, in order of increasing eigenvalue, are in the directions given by

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

Clue G: the matrix $3X$ has integer elements.

Show that the clues A, B, C and D cannot all be true together.

Show that the two clues E and F cannot be true together.

Level 4

You are now told that G is true, three of the clues A, B, C and D are true, and one from the pair E and F is true.

What are the possibilities for X?

You can use your calculator to find matrix inverses and for multiplying matrices.

The Notes for this risp are on page 132; Home Page

Further Risp 8: Induction Number Theory

Level 1

What do you understand by the phrase 'proof by induction'?

Can you prove using induction that 6 divides into $n^3 - n$ for all $n \geq 0$?

The useful notation $a|b$ means 'a divides b', so you need to prove

$$6|n^3 - n \quad \text{for all} \quad n \geq 0.$$

Level 2

Can you prove by induction that $7^n + 4^n + 1$ is divisible by 6 for all $n \geq 1$?

Level 3

Can you generalise this? When is $a^n + b^n + c$ divisible by d for all positive n? (You could call this a d-conjecture.)

Could you write a small computer program, perhaps within an Excel spreadsheet, to search for possible d-conjectures for small a, b, c and d?

For advice on writing simple computer programs in Visual Basic within Excel, see page 103.

A spreadsheet incorporating a possible program is at **www.further-risps.co.uk**.

Level 4

Your computer program might suggest these possible d-conjectures; can you prove or disprove these possibilities?

$$8|3^n + 7^n + 6 \quad \text{for all} \quad n \geq 0.$$

$$9|4^n + 7^n + 7 \quad \text{for all} \quad n \geq 0.$$

Level 5

By using a computer, can you come up with a list of d-conjectures of your own that you could try to prove using induction?

The Notes for this risp are on page 134; Home Page

Further Risp 9: The Unexpected Group

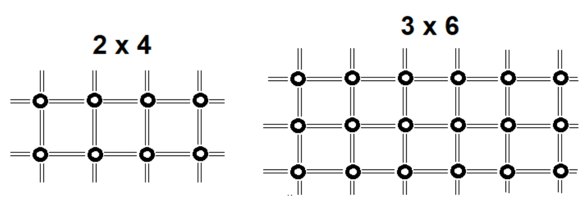
Level 1

What do we mean by 'the natural numbers'? By 'the integers'? By 'the rational numbers'? By 'the real numbers'? What symbols do we use for these sets of numbers?

What do mathematicians mean by 'a group'?

Level 2

The word 'multiply' can mean all sorts of things. You can draw a rectangle on a grid like this...



This gives you the 'normal' definition of multiplication, let's call it $\times$, for the natural numbers. We can extend this definition to the integers, to the rational numbers and to the real numbers.

Is the set of natural numbers together with $\times$ a group?

Is the set of integers together with $\times$ a group?

Is the set of rational numbers together with $\times$ a group?

Is the set of real numbers together with $\times$ a group?

Level 3

Now tilt your rectangle, as in Figure A.

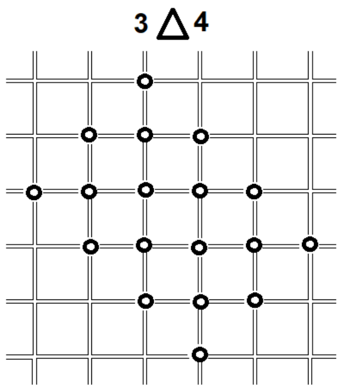


Figure A

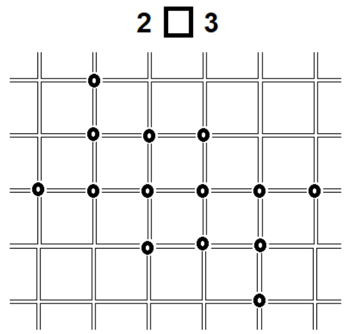


Figure B

Here you have a new binary operation, say $\triangle$, where $x \triangle y$ is the number of grid-points coloured in the diagram. Just as you did before, you can extend this idea of multiplication beyond the natural numbers.

Is the set of natural numbers together with $\triangle$ a group?

Is the set of integers together with $\triangle$ a group?

Is the set of rational numbers together with $\triangle$ a group?

Is the set of real numbers together with $\triangle$ a group?

Level 4

What happens if you tilt our rectangle differently, as in Figure B? Can $\square$ be the binary operation for a group?

Is it possible to generalise this for a range of tilts?

The Notes for this risp are on page 140; Home Page

Further Risp 10: First Order Differential Equations

Level 1

Suppose you face the differential equation $\dfrac{dy}{dx} + Py = Q$, where P and Q are functions of x.

You decide to try to solve this by multiplying both sides by R, a function of x, so that you turn the left-hand side of the equation into the exact derivative of a product.

What function for R is the sensible one to choose? Can you show this?

(This is known as the *integrating factor* method for solving the differential equation.)

Level 2

Success here is not guaranteed; you may still be stuck if either P or Q is a function that makes life difficult!

Find a pair of functions P and Q so that solving the differential equation $\dfrac{dy}{dx} + Py = Q$ exactly using the integrating factor method

1. is easy.
2. is difficult but possible.
3. is impossible.

Level 3

Staying with the integrating factor method, what happens if $P = Q$?

Pick a simple function for P, and try to solve the differential equation

$$\frac{dy}{dx} + Py = P$$

using an integrating factor.

Now pick another function for P, and another; how hard is the differential equation to solve each time?

What happens if you try to solve $\dfrac{dy}{dx} + Py = P$ in general using an integrating factor?

Level 4

Is there an alternative way to solve the differential equation $\dfrac{dy}{dx}+Py = P$?

Does it give the same answer?

The Notes for this risp are on page 144; Home Page

Further Risp 11: Characteristic Equations

Level 1

You are told that $t^2 + 3t - 4 = 0$ is the characteristic equation of the matrix $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$. Can you find a and b? Is the answer unique?

Is it always possible for $t^2 + pt + q = 0$ (where p and q are real) to be the characteristic equation of a matrix of the form $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ (where a and b are real)? How many choices for a and b are there each time?

Level 2

How does this work in 3 dimensions?

Can you find a 3×3 matrix with real elements of the form $\begin{pmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ that has $t^3 + pt^2 + qt + r = 0$ as its characteristic equation?

How many choices for a, b and c are there for a given choice of p, q and r?

Level 3

If you are given three real numbers α, β and γ, can you swiftly find a 3×3 matrix that has these values as its eigenvalues?

Can you find a matrix of the form $\begin{pmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ with the eigenvalues α, β and γ?

Level 4

What are the eigenvectors of the matrix $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$? Of the matrix $\begin{pmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$?

Can you comment on these results?

The Notes for this risp are on page 146; Home Page

Further Risp 12: Sketching a Rational Function

You are given the following equation of a curve, where a, b, c and d are all rational numbers ($d \neq 0$):

$$y = \frac{ax^2 + bx + c}{x^2 + x + d}$$

Pick three distinct integers in the range $0 < x < 10$.

Now swap these with the three question-marks below in some order (no repeats!)

The curve has

1. *a turning point with an* x*-coordinate of 0*
2. *a vertical asymptote at* $x =$ *?*
3. *a horizontal asymptote at* $y =$ *?*
4. *an* x*-intercept at (?, 0)*

Can you find values for a, b, c and d that make these statements truthful?

Once you have your answer, check it with a graph plotter.

Will any starting choice of integers work, or are there some restrictions?

Why do you need to exclude the case $d = 0$?

The Notes for this risp are on page 148; Home Page

Further Risp 13: The Rotating Triangle

Level 1

In polar coordinates, the point $A = (r, \theta)$ is a distance r from the origin, with OA making an angle θ with the principal line.

Plot the three points A, B and C below on a computer graphing page. You may wish to convert these points into their Cartesian representation (x, y).

Point	r	θ	x	y
A	1	0		
B	$\sin a$	a		
C	1	$2a$		

Your plotter should put $a = 1$ *to start with.*

Join A, B and C with three line segments to create a triangle. Add the circle $r = 1$ to the diagram. Now vary a; what do you see?

If $\dfrac{da}{dt} = k$ where k is constant, how fast are A, B and C travelling?

Level 2

What kind of a triangle is this? Is it ever equilateral?

What is the locus of B? When is the origin inside the triangle?

Level 3

When is the area of this triangle a maximum?

Is the triangle for which the area is a maximum equilateral? Can you prove this, or disprove it?

Level 4

Now replace the point B with D, where $D = (\cos\theta, \theta)$ in polar coordinates.

Describe the motion of the triangle ADC. Can you explain this?

What is the locus of the point D as a varies?

Level 5

Now replace A, D and C with E, F and G, as follows;

Point	r	θ
E	$\cos a$	0
F	$\tan a$	a
G	$\cos a$	$2a$

Examine what happens to the triangle EFG as a varies.

Does the area of this triangle have a maximum value? If so, what is it?

The Notes for this risp are on page 151; Home Page

Further Risp 14: Three Simultaneous Equations

You will need access here to a calculator that will allow you to manipulate 3×3 *matrices.*

Level 1

What do we mean when we say a square matrix is singular? Are there ways to tell quickly if a square matrix is singular?

Level 2

Imagine that you have three equations in three unknowns x, y and z to solve.

$$\square x + \square y + \square z = \square$$
$$\square x + \square y + \square z = \square$$
$$\square x + \square y + \square z = \square$$

What happens if you fill the squares with consecutive terms taken from a sequence? For example, you could choose the sequence

$$1, 2, 3, 1, 2, 3, 1, 2, 3...$$

$$\boxed{1}\,x + \boxed{2}\,y + \boxed{3}\,z = \boxed{1}$$

$$\boxed{2}\,x + \boxed{3}\,y + \boxed{1}\,z = \boxed{2}$$

$$\boxed{3}\,x + \boxed{1}\,y + \boxed{2}\,z = \boxed{3}$$

What's the solution here? Can you test out other sequences $a, b, c, a, b, c, ...$?

Can you make a conjecture? Can you prove it?

Level 3

What happens if you choose 12 consecutive terms from an arithmetic sequence? Can you generalise starting from concrete examples?

What happens if you choose 12 consecutive terms from a geometric sequence? Can you generalise?

Level 4

Define a Fibonacci-style sequence as one where you get the next term by adding the previous n terms together. You can choose your first n terms freely!

What happens if you choose 12 consecutive terms from a Fibonacci-style sequence when $n = 3$? Can you generalise?

Level 5

Can you think of any sequences of your own to try?

The Notes for this risp are on page 154; Home Page

Further Risp 15: Matrix Logic

Below you are given twelve pairs of statements A and B, each involving either P or Q or both, where P and Q are square matrices of equal order with real elements.

In each case, say whether one statement implies the other (for example, $A \Rightarrow B$) or does not imply the other (for example, $B \not\Rightarrow A$). You may be able to write $A \Leftrightarrow B$. It may also be the case that one statement rules out the other. It could also be that one or both statements are always true, or always false.

You should tackle this for the cases when P and Q are either both 2×2 or both 3×3 matrices.

Level 1

1. $A : \det(P) = \det(Q) = k, k \in \mathbb{Z}$,
$B : \det(PQ)$ is a perfect square.

2. $A : P$ represents a rotation about O,
B (for 2×2) $: P$ represents a reflection in a line through the origin.
B (for 3×3) $: P$ represents a reflection in a plane through the origin.

3. $A : PQ$ is singular,
$B : P$ or Q or both are singular.

Level 2

4. $A : PQ = QP$,
$B : P$ and Q both represent rotations about the origin.

5. P is not singular.
$A : \det(kP) = k \det(P)$,
$B : k = 0$ or 1.

6. $A : P$ represents a rotation about the origin,
$B : \det(P) = 1$.

Level 3

7. $A : P$ represents a rotation through θ about the origin, $0 < \theta < \pi$,
$B : P$ has no real eigenvectors.

8. $A : P$ is the zero matrix,
$B : P$ has no real eigenvalues.

9. P and Q are not singular.
$A : \det(P + Q) = \det(P) + \det(Q)$,
$B : P = Q$.

Level 4

P and Q are not singular.
10. $A : (PQ)^{-1} = P^{-1}Q^{-1}$,
$B : PQ = QP$.

11. P and Q are 2×2 reflection matrices.
$A : PQ = QP$,
$B : P = Q$.

12. T is an enlargement with scale factor $k \neq 1$.
$A : T$ can be represented by a matrix P,
$B : T$ has its centre at the origin.

The Notes for this risp are on page 158; Home Page

Further Risp 16: Periodic Recurrence Relations

Level 1

Consider the following recurrence relation;

$$u_{n+2} = -u_{n+1} - u_n, n \geq 1 \quad \text{with} \quad u_1 = 1, u_2 = 2.$$

So the next term of the sequence is the negative of the sum of the previous two. This gives you this sequence $1, 2, -3, 1, 2, -3, 1, ...$ (check this!)

Since this sequence repeats every three terms, you can say it is *periodic*, with period 3.

You can write this recurrence simply as $z = -y - x$, where $z = u_{n+2}$, $y = u_{n+1}, x = u_n$ and where the starting values are also given. This notation for defining recurrences turns out to be extremely convenient.

You can write your sequence above in general as

$$x, y, -y - x, x, y, -y - x, x, ...$$

where the starting values now are x and y. Can you find the next few terms of the recurrence $z = -y - x - w$, where the terms begin $w, x, y, z, ...$?

Level 2

You can call the recurrence $z = -y - x - w$ order-3, since the next term is defined in terms of the previous three. You can define recurrences of any positive order, including order-1; here's an example:

$$u_{n+1} = \frac{u_n + 1}{u_n - 1}, n > 0 \quad \text{with} \quad u_1 = 2.$$

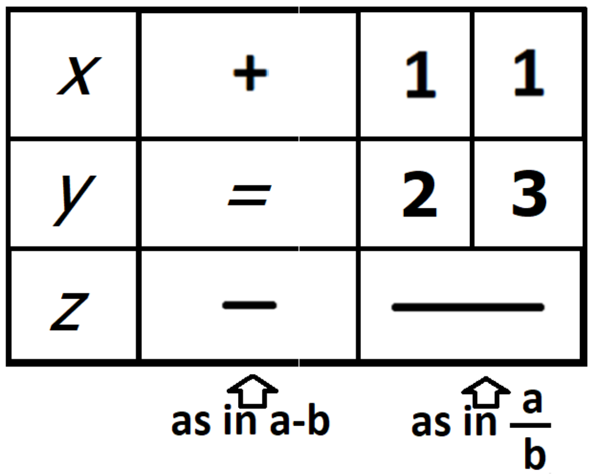
You can write this in the more convenient form as

$$y = \frac{x + 1}{x - 1} \quad \text{with first term} \quad 2.$$

Can you find the first few terms of this recurrence? What if instead your starting term is x?

Level 3

The three recurrences examined so far have all been periodic, for almost all starting values. This behaviour is in fact very unusual, and interesting! You are now given this set of cards to explore this periodic behaviour further.



One of these cards represents subtraction, and another represents division.

Use some or all of these cards (no repeats!) to define an order-1 or order-2 recurrence.

Assume your starting terms are x and y for an order-2 recurrence $z = f(x, y)$, and x for an order-1 recurrence $y = g(x)$.

What different types of behaviour can you find? Periodic behaviour will be of especial interest.

An Excel spreadsheet can be helpful for computing further terms. You could download the spreadsheet for order-1 and order-2 recurrences at

www.further-risps.co.uk

Level 4

Hard!

If a, b, c, d are all integers, what periods are possible for the order-1 recurrence

$$u_{n+1} = \frac{au_n + b}{cu_n + d}, n > 0 \quad \text{with} \quad u_1 = x?$$

If a, b, c, d, e, f are all integers, what periods are possible for the order-2 recurrence

$$u_{n+1} = \frac{au_n + bu_{n-1} + c}{du_n + eu_{n-1} + f}, n > 1 \quad \text{with} \quad u_1 = x, u_2 = y?$$

You can use the Excel spreadsheet to help you.

The Notes for this risp are on page 163; Home Page

Further Risp 17: Summing Powers

Level 1

You are given that $\sum_{1}^{n} ar + b = An^2 + Bn$ for all positive integers n.

You might ask these questions;

Given any a and b, can I always find A and B?
Given any A and B, can I always find a and b?
If a and b are integers, will A and B always also be integers?
If A and B are integers, will a and b always also be integers?

Level 2

You are now given that $\sum_{1}^{n}(cr + d)r = Cn^3 + Dn^2 + En$ for all positive integers n.

What happens if you ask the same kind of questions?

Level 3

You are now given that $\sum_{1}^{n}(fr + g)(r + 1) = Fn^3 + Gn^2 + Hn$ for all positive integers n.

What happens if you ask the same kind of questions?

Level 4

You are now given that $\sum_{1}^{n}(jr + k)(r + 1)r = Jn^4 + Kn^3 + Ln^2 + Mn$ for all positive integers n.

What happens if you ask the same kind of questions?

The Notes for this risp are on page 165; Home Page

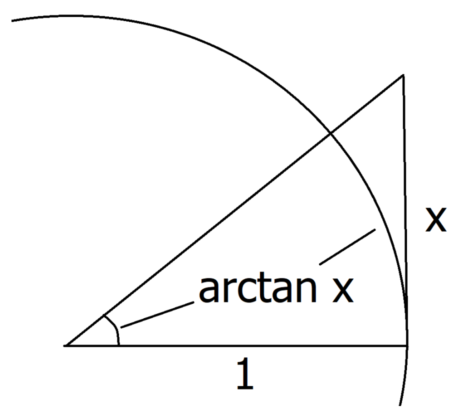
Further Risp 18: Do you really mean 'arctan'?

What name do we give the function, 'the angle whose tan is x'?

In 1816, the astronomer John Herschel suggested we use $\tan^{-1} x$ for this function, the inverse of $\tan x$, in the same way that we use f^{-1} for the inverse of f.

But isn't this potentially confusing? If by $\tan^2 x$ we always means $(\tan x)^2$, then surely $\tan^{-1} x = \dfrac{1}{\tan x}$?

Why do we add the prefix 'arc' to the name 'arctan'? Since for the unit circle, 'the angle whose tan is x' is the same as 'the arc whose angle's tan is x'.



Maybe there are values of x for which confusing $\dfrac{1}{\tan x}$ with $\arctan x$ doesn't matter - we get the same answer either way!

This would mean that $\dfrac{1}{\tan x} = \arctan x$, which is an impossible equation to solve algebraically. In cases such as these, we'll need to use an approximate numerical method.

Level 1

You will need to use a computer graphing package for this task.

Does $\dfrac{1}{\tan x} = \arctan x$ have roots? If so, can you find the positive root closest to the origin using the rearrangement method (the $x = g(x)$ iteration)?

Level 2

Does $\dfrac{1}{\sin x} = \arcsin x$ have roots? If so, can you use the Newton-Raphson method to find the positive root closest to the origin?

Level 3

Does $\dfrac{1}{\cos x} = \arccos x$ have roots? If so, can you try linear interpolation here? Again, it's the positive root closest to the origin you need to find.

Level 4

Are there any solutions for

$$\arccos x + \arcsin x + \arctan x = \frac{1}{\cos x} + \frac{1}{\sin x} + \frac{1}{\tan x}?$$

The Notes for this risp are on page 168; Home Page

Further Risp 19: Roots of Cubic Equations

Level 1

Suppose that $ax^3 + bx^2 + cx + d = 0$ has the roots α, β and γ. What can you say about the sum of the roots $\alpha + \beta + \gamma$, the product of the roots $\alpha\beta\gamma$ and the sum of the product of pairs of roots $\alpha\beta + \gamma\alpha + \beta\gamma$?

Level 2

How would you solve the following equations exactly? (Some roots may be complex.)

You only need to give your method, you don't need to actually find the roots. You may need to do some research on the internet.

1. $x + 1 = 0$?

2. $x^2 + x + 1 = 0$?

3. $x^3 + x^2 + x + 1 = 0$?

4. $x^3 + x^2 + x + 2 = 0$?

5. $x^3 + x^2 + x = 0$?

Level 3

6. $x^4 + x^3 + x^2 + x + 1 = 0$?

7. $x^4 + x^2 + 1 = 0$?

8. $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$?

9. $x^5 + x^4 + x^3 + x^2 + x + 2 = 0$?

Level 4

Pick four non-zero integers $p, q, r, s \geq 2$ with no common factor shared between all four so that

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix} = 0.$$

You are told the equation $px^3 + qx^2 + rx + s = 0$ has the roots α, β and γ, and your task is to find these exactly.

Write your equation down for your choice of numbers; how could you solve this to find α, β and γ?

One possible method is suggested here;

Can you find the equation with roots $\alpha + \beta, \beta + \gamma, \gamma + \alpha$?

Can you solve this equation, and from here find α, β and γ?

Does this method always work? Can you prove it?

Level 5

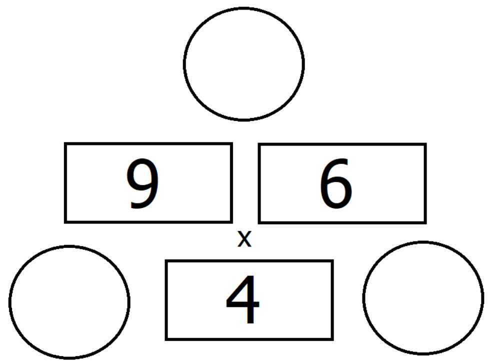
Looking at your answers, can you find an alternative way to solve

$$px^3 + qx^2 + rx + s = 0 \quad \text{when} \quad ps - qr = 0?$$

The Notes for this risp are on page 173; Home Page

Further Risp 20: Matrix Arithmagon

Level 1



This is a multiplication arithmagon. The numbers in a pair of circles multiply to the number in the rectangle between them.

Can you find the numbers in the circles here? How many solutions are there?

Level 2

If square matrices A and B are such that $A^2 = B$, then A is a *square root* of B.

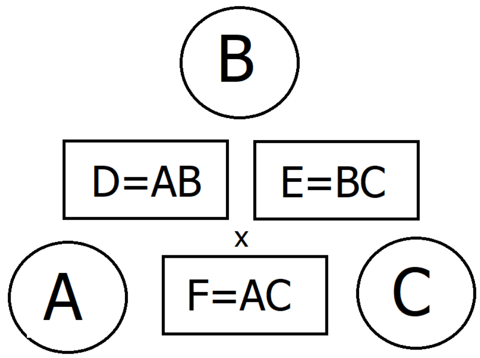
For 2×2 matrices, how many square roots would you expect a matrix to have in general?

If a matrix A has determinant k, what are the possible values for the determinant of a square root of A?

Can you find the square roots of the matrix $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$?

Hold on to your answers...

Level 3



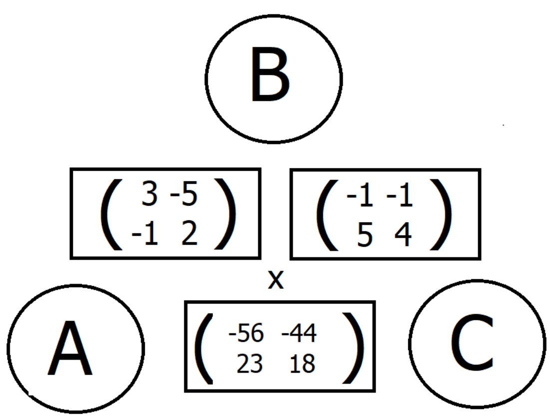
In the above arithmagon, A, B, C, D, E and F are all 2×2 matrices.

Given that D, E and F are all non-singular, what strategy would you use to find A, B and C? Are you making any assumptions here?

How many solutions for A, B and C do you think there are?

Beware! Matrix multiplication is not commutative.

Level 4



Find all possible solutions for A, B and C in the arithmagon above.

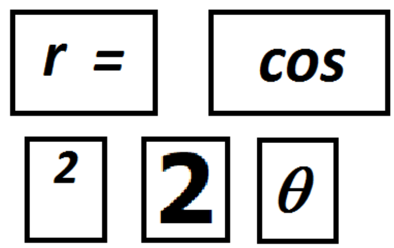
The Notes for this risp are on page 176; Home Page

Further Risp 21: Ordering Loops

Tackle this without a computer or a calculator to start with, and then use one at the end to confirm your answers.

Level 1

You're given the following set of cards;



The card with the small 2 at the top always means 'square'.

How many legal polar coordinate equations can you make using ALL of these cards but no others? *No repeats!*

The cards have to lie on a line; only the 'square' card can be used as a power.

You can also use as many brackets as you like.

The equals sign must follow the r, but cards can come before the '$r =$' card.

Level 2

Can you categorise your curves? Which of your equations represent spirals? Straight lines? Which are circles? Which are rose curves (with petals)?

Which curves are hard to categorise?

Level 3

Some of your equations will give a symmetrical loop that runs across the first and fourth quadrants, with the principal line as a line of symmetry.

Can you choose five of your equations that give such a loop, and try to order them by loop size?

If there's an order you're not sure about, you may have to do some calculations to resolve this.

Level 4

Now check your answers using a computer graphing package or a graphical calculator.

Finding the area of some of these loops exactly is relatively straightforward; for which of your equations is this possible? Give your methods but not your exact answers.

How could you find the area of a loop approximately?

The Notes for this risp are on page 179; Home Page

Further Risp 22: The Vector Product

Level 1

You are given two vectors $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ where $\underline{b}$ is not a multiple of $\underline{a}$.

You are told the vector $\underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ is perpendicular to both $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Using the scalar product, can you find a vector in the direction of $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ in terms of a_1, a_2, a_3, b_1, b_2 and b_3?

Can you show this vector must be a multiple of $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$?

Level 2

The vector product of two vectors $\underline{a}$ and $\underline{b}$ can be defined like this;

$$\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin\theta \underline{\hat{n}},$$

where $\underline{\hat{n}}$ is a unit vector perpendicular to the plane of $\underline{a}$ and $\underline{b}$ in a right-hand screw sense, and where θ is the angle between $\underline{a}$ and $\underline{b}$.

Fact: $\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

Now pick vectors $\underline{a}$ and $\underline{b}$ so that the elements a_1, a_2, a_3, b_1, b_2 and b_3 consist of three positive and three negative numbers.

if $\underline{a} \times \underline{b} = \underline{c}$, can c_1, c_2 and c_3 all be positive?

Level 3

If you pick six positive numbers for a_1, a_2, a_3, b_1, b_2 and b_3, is it possible for c_1, c_2 and c_3 to all be positive too?

Level 4

Imagine the planes $x = 0, y = 0$ and $z = 0$ in three dimensions. Into how many regions do these planes divide three-dimensional space? How could you label these regions sensibly ?

If a_1, a_2, a_3, b_1, b_2 and b_3 are all positive, in which of these regions can the position vector $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ lie?

The Notes for this risp are on page 182; Home Page

Further Risp 23: The Matrix Puzzle

Level 1

Knowledge check;

Given a 2×2 or a 3×3 matrix, can you say what its trace is? Its determinant?

Can you find its eigenvalues? Its eigenvectors?

What is its characteristic equation?

How are the eigenvalues and the trace related?

How are the eigenvalues and the determinant related?

Level 2

The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a < d$ and $b < c$, has trace t and determinant u.

Its eigenvalues are e and f, where $e < f$, and examples of corresponding eigenvectors are

$$\begin{pmatrix} p \\ q \end{pmatrix} \text{ and } \begin{pmatrix} r \\ s \end{pmatrix}.$$

The values for $a, b, c, d, e, f, p, q, r, s, t$ and u are (ordered by size)

$$-403, -85, -13, 4, 9, 14, 18, 30, 31, 45, 51, 90.$$

Which value goes with which letter?

Now for a 3×3 version...

Level 3

The matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ d & e & f \end{pmatrix},$$

has trace t and determinant u.

Its eigenvalues are $g < h < i$, and examples of corresponding eigenvectors are

$$\begin{pmatrix} j \\ k \\ l \end{pmatrix}, \begin{pmatrix} m \\ n \\ p \end{pmatrix} \text{ and } \begin{pmatrix} q \\ r \\ s \end{pmatrix}.$$

You are told $b < f, k < l, n < p, r < s, j = 0$ and $p = 2680$.

The values for $a, b, c, d, e, f, g, h, i, k, l, m, n, q, r, s, t$ and u are (ordered by size)

$$-2880, -2412, -51, -22, -7, -6, 1, 3, 7, 10, 11, 15, 16, 21, 51, 134, 270, 4500.$$

Can you find which value goes with which letter?

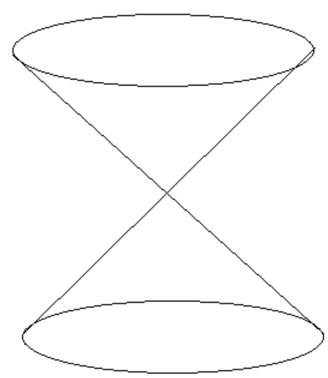
The Notes for this risp are on page 186; Home Page

Further Risp 24: One-Parameter Conics

Level 1

The conics are a family of curves that can be defined in several ways.

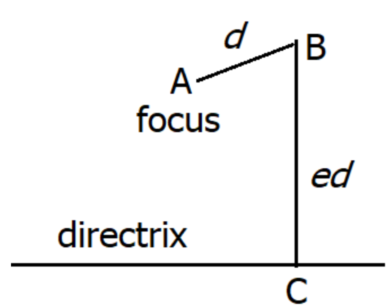
1. Take a double cone like the one below. Make a plane cut; what are the possible curves of cross-section?



2. Take a line (the *directrix*) and a point A (the *focus*) and ask, what is the locus of a point B moving so that

$$AB = e \times \text{the distance of} \quad B \quad \text{from the directrix?}$$

(The value e is a non-negative constant for the curve called the *eccentricity.*)



3. What curves can you generate as you vary a, b, c, d, e, f in the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$?

Investigate each of these three situations, and see what curves you can find. Can you find connections between these different definitions?

Level 2

In a graphing package, plot the curve $kx^2 + y^2 + x = 1$.

Now vary k between $-\infty$ and $+\infty$; what curves are possible?

Level 3

How does your list compare with your double cone list above?

Are there any curves missing? Or can you make any extra curves?

Is every eccentricity possible? Can you make all conics this way, to within similarity?

Level 4

Can you come up with any arguments to support your conjectures?

The Notes for this risp are on page 189; Home Page

Further Risp 25: Trace Arithmetic

Level 1

The trace of a square matrix is defined as $\text{tr}(A) =$ the sum of the elements on the main diagonal, so $\text{tr}\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix} = 6$.

Pick two 2×2 matrices $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $P = \begin{pmatrix}p & q\\ r & s\end{pmatrix}$.

What is $\text{tr}(AP)$? What is $\text{tr}(PA)$? What's the connection here? Can you prove this always works?

Level 2

Does this rule work for 3×3 matrices? *You can use a calculator to test cases.*

Can you make a conjecture? Can you prove it?

Level 3

Define A^H to be the reflection of the square matrix A in its central horizontal line, so if $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ then $A^H = \begin{pmatrix}c & d\\ a & b\end{pmatrix}$.

Similarly define A^V to be the reflection of the square matrix A in its central vertical line, so if $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ then $A^V = \begin{pmatrix}b & a\\ d & c\end{pmatrix}$.

Pick four values for a 2×2 matrix A, find $\text{tr}(A), \text{tr}(A^H), \text{tr}(A^H A)$ and $\text{tr}(AA^H)$. How are these connected?

Is there a similar result for A^V?

Does this connection hold for any matrices other than A^H and A^V?

Level 4

Is there a similar connection for 3×3 matrices? *You can use a calculator to test cases.*

The Notes for this risp are on page 195; Home Page

Further Risp 26: The Coefficient Function

Using mathematical symbols can be most economical! They can pack a lot of question into a small space. Here's an example...

The Maclaurin expansion of $f(x)$ is found by writing $f(x)$ as the sum of multiples of the non-negative powers of x.

This expansion may only be valid for a limited range of x.

Define $f_a(x)$ as 'the coefficient of x^a in the Maclaurin expansion of $f(x)$.'

You are now given that

$$f(x) = e^x, g(x) = \ln(1 + x), h(x) = \cos x, i(x) = \sin x.$$

In the questions below, p, q and r are all positive integers.

Level 1

What is $f_3(x)$? What is $g_4(x)$?

Level 2

Can you find integers p and q so that $f_3(px) + g_4(qx) = 0$?

If so, for which values of x is your answer valid?

Level 3

Can you find a solution to $ph_r(qx) = qi_r(px)$?

Can you find a solution to $-5ph_q(rx) = rg_p((q - 2)x)$ when $p = 3$ and $q = 6$?

Level 4 *Hard!*

Can you find all possible solutions to $|pg_r(qx)| = |qh_p(rx)|$ when $p = 2$?

The Notes for this risp are on page 197; Home Page

Further Risp 27: Hyperbolic Trigonometry

Level 1

Let's define

$$C_1(x) = \cos x \cosh x, C_2(x) = \cos x + \cosh x,$$

and

$$S_1(x) = \sin x \sinh x, S_2(x) = \sin x + \sinh x.$$

Are these four functions odd or even or neither?

Level 2

What is $\dfrac{d}{dx}S_2(x)$ in terms of these functions? What is $\dfrac{d}{dx}S_1(x)$?

Can you write $S_1(2x)$ in terms of $C_1(x)$ and $S_1(x)$?

Can you find an identity for $C_1(2x)$ in terms of these four functions?

Level 3

You are told that two curves $y = f(x)$ and $y = g(x)$ touch at $x = k$ if and only if

$$1.\ f(k) = g(k), \text{ and } 2.\ f'(k) = g'(k).$$

Why should this be the case? Are there any exceptions to this rule?

Use this result to show that the curve $y = x^3 + 4x^2 - 10x + 4$ touches the curve $y = x^2 + 14x - 24$ at $x = 2$.

Level 4

Try to sketch the graphs

$$y = C_1(x), y = C_2(x), y = S_1(x), y = S_2(x).$$

Using a computer graphing package, can you confirm your sketches?

Are there any regions where two curves seem to touch?

How could you decide whether these points are genuine touching points or near misses?

Try to prove any conjectures you make.

The Notes for this risp are on page 200; Home Page

Further Risp 28: Factorial Sums

Some series involving the factorial function are very hard to sum.

Take, for example, $\sum_{r=1}^{n} r!$ – the internet tells us that this sum can be found, true, but the maths is fiendish!

What happens if you try to find instead $\sum_{r=1}^{n} f(r)r!$ where $f(r)$ is a polynomial in r? Could a sensible choice for $f(r)$ make things easier?

Level 1

What is $\sum_{r=1}^{n} [(r+1)! - r!]$?

Can you write this as $\sum_{r=1}^{n} f(r)r!$ for some $f(r)$?

Level 2

Can you explain why trying to find $\sum_{r=1}^{n} (r+1) \times r!$ or $\sum_{r=1}^{n} (r-1) \times r!$ will not work in the same way? What happens to the factorials here?

In what form do you need to write $f(r)$ in order to create factorials?

If you try $f(r) = ar + b$, what conditions are there now on a and b for this 'cancelling factorial' method to work?

Level 3

What if $f(r)$ is quadratic? If $f(r) = ar^2 + br + c$, for what values of a, b and c does this method succeed?

Level 4

Is it possible to generalise? If $f(r)$ is of degree n, and this method succeeds, what can you say about the coefficients of the powers of r in $f(r)$?

You could use a computer algebra package to help you here.

If you want to check an integer sequence, try

https://oeis.org/,

the Online Encyclopedia of Integer Sequences.

The Notes for this risp are on page 204; Home Page

Further Risp 29: The Same Direction

Level 1

You're given two non-zero complex numbers, w and z. You are told this; if you

1. add w to z, and then 2. multiply the result by w,

the result is a complex number in the same direction as z, say jz, where j is real.

This is fact also true; if you

1. multiply z by w and then 2. add w to the result,

you again get a complex number in the same direction as z, say kz, where k is real.

Can you write down two equations that z and w must satisfy?

Can you solve your two equations for z and w simultaneously?

Level 2

Suppose now that $k = 1$ and $j > 1$, say $j = 1 + m^2$, where m is real and positive.

What can you now say about w and z?

What must the angle between z and w be? Can you draw a diagram to show the solutions?

With these same values for j and k, on which curves do w and z lie?

What can you say about $|z - w|$? When is this a minimum?

Level 3

Hard!

Now suppose instead that $j = 1$, and $0 < k < 1$.

What can you now say about w and z?

On which curves do w and z lie now?

Draw the loci of w and z for $0 < k < 1$.

The Notes for this risp are on page 208; Home Page

Further Risp 30: A Matrix with Unit Elements

Level 1

Given a 2×2 matrix, how would you define the trace? The determinant?

			Trace	
		Less	*Same*	*More*
Determinant	*Less*			
	Same		$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$	
	More			

Can you find a matrix for each gap in the diagram above?

Note; you are only allowed to use -1, 0 and 1 for the elements of your 2×2 *matrix.*

What are the maximum/minimum possible values for the trace? For the determinant?

Level 2

What happens if you multiply every element of every matrix by 2; does the table still hold true?

What happens if instead you multiply every element of every matrix by -2; does the table still hold?

Suppose you now increase every element of all the original matrices by 1, so that each matrix now contains only zeroes, 1s and 2s. Does the table still remain true?

Level 3

Given a 3×3 matrix, how would you define/calculate the trace? The determinant?

Determinant \ Trace	Less	Same	More
Less			
Same		$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$	
More			

Can you find a matrix for each gap in the diagram above?

Note; you are only allowed 3×3 *matrices consisting of two* 1*s, two* -1*s, and five zeroes. No calculators!*

What are the maximum/minimum possible values for the trace? For the determinant? Can you prove it?

Level 4

What happens if you multiply every element of every matrix by 2; does the table still hold true?

What happens if instead you multiply every element of every matrix by -2; does the table still hold?

Suppose you now increase every element of all the original matrices by 1, so that each matrix now contains two zeroes, five 1s and two 2s. Does the table still remain true?

The Notes for this risp are on page 213; Home Page

Further Risp 31: Matrix Segments

Level 1

Pick six points in the $x-y$ plane (not the origin), and call them A, B, C, D, E and F, where
$A = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, B = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, C = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}, D = \begin{pmatrix} x_4 \\ y_4 \end{pmatrix}, E = \begin{pmatrix} x_5 \\ y_5 \end{pmatrix}, F = \begin{pmatrix} x_6 \\ y_6 \end{pmatrix}.$

Draw the triangles ABC and DEF. Can you find a matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ representing a linear transformation that maps the line segment AB to the line segment DE?

Is this always possible? Are there any conditions on A, B, C and D for this to work?

If this is impossible for your choice of numbers, adjust your numbers so that it is possible.

Level 2

Can you find a matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ representing a linear transformation that maps your line segment AB to your line segment DE after DE has been translated through a vector $\begin{pmatrix} t \\ u \end{pmatrix}$ for some t and u?

Give your answers in terms of t and u.

Level 3

Would you expect to be able to find a matrix that maps your triangle ABC exactly onto your triangle DEF?

Level 4

Using your Level 3 work, can you find a matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ that maps your triangle ABC exactly onto a translation of your triangle DEF through a vector $\begin{pmatrix} t \\ u \end{pmatrix}$ for some t and u?

Is finding values for p, q, r, s, t and u always possible here, whatever your original choices for A, B, C, D, E and F?

The Notes for this risp are on page 217; Home Page

Further Risp 32: Binary Operation Venn Diagram

Level 1

Given a set of objects S, a binary operation o on S is a way of combining two elements of S, to give something that may or may not be in S.

So if $\mathbb{Q}$ is the set of rational numbers, $+$ (where $+$ is our standard idea of addition) is a binary operation on $\mathbb{Q}$.

We can see that if x and y are in $\mathbb{Q}$, then $x+y$ is always in $\mathbb{Q}$ too. We say $+$ is **closed** here.

We can also see that if x, y and z are in $\mathbb{Q}$, then $x+(y+z) = (x+y)+z$; we say $+$ is **associative** here.

Also, if x and y are in $\mathbb{Q}$, then $x+y = y+x$; we say $+$ is **commutative** here.

If the set $\mathbb{Q}$ together with $+$ is to be a group, what extra facts need to be in place? Are the above three properties essential for $(\mathbb{Q}, +)$ to be a group?

Level 2

$+$	$\mathbb{N}$	$\mathbb{Z}$
Associative		
Commutative		
Closed		

$-$	$\mathbb{N}$	$\mathbb{Z}$
Associative		
Commutative		
Closed		

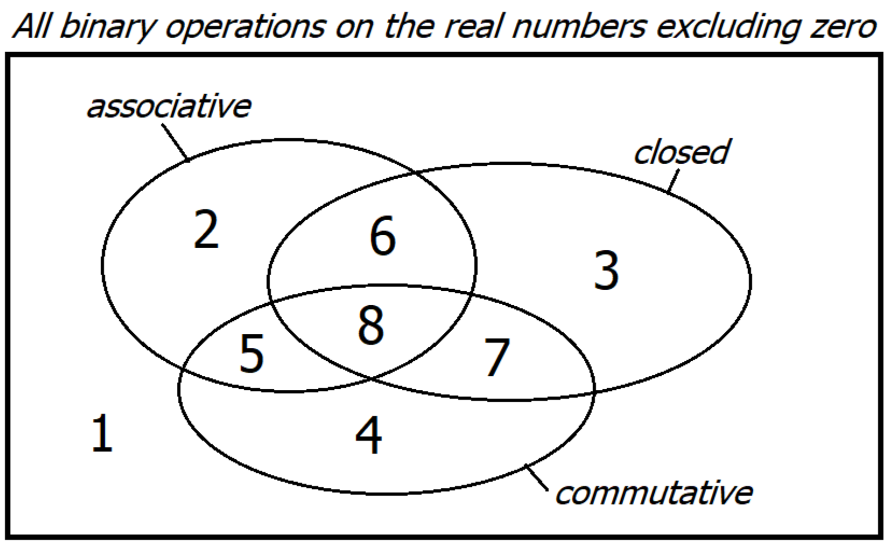
Should the boxes above contain a tick or a cross?

Here $\mathbb{N}$ is the set of natural numbers {0, 1, 2, 3...} and $\mathbb{Z}$ is the set of integers {..., -2, -1, 0, 1, 2....}

Level 3

Can you find binary operations for each of the eight regions as numbered in the Venn diagram below?

Warning; some regions may be impossible!



The set the box contains could be written as $S = \mathbb{R}/\{0\}$.

The numbers in the regions here are simply numbering the regions for convenience.

The Notes for this risp are on page 221; Home Page

Further Risp 33: Adding Three Integrals

Level 1

What is $\int \frac{1}{1+x^2}dx$?

What is $\int \frac{1}{a^2+x^2}dx$?

What is $\int \frac{b^2}{a^2+c^2x^2}dx$?

Level 2

Suppose a, b, c, d, e are positive real numbers that are also consecutive terms from a geometric progression with common ratio r.

What is $I_1 = \int_d^e \frac{b}{a+cx^2}dx$?

Level 3

Can you say from this what $I_2 = \int_e^a \frac{c}{b+dx^2}dx$ is?

How about $I_3 = \int_a^b \frac{d}{c+ex^2}dx$?

Level 4

Can you find simple values for p and q in terms of b, d and r so that

$$\int_p^q \frac{c}{a+ex^2}dx = I_1 + I_2 + I_3?$$

The Notes for this risp are on page 224; Home Page

Further Risp 34: The Series Result

Level 1

Can you write $\dfrac{1}{(r+4)(r+5)}$ as partial fractions? What is $\displaystyle\sum_{1}^{100} \frac{1}{(r+4)(r+5)}$?

Let's define $S_{a,b,n} = \displaystyle\sum_{1}^{n} \frac{1}{(r+a)(r+b)}$.

Pick a positive whole number $a > 5$, and find the values of the three expressions

1. $S_{a,a+1,100}$
2. $S_{a+1,a+2,100}$
3. $S_{a,a+2,100}$

What do you find if you calculate $\dfrac{S_{a,a+1,100} + S_{a+1,a+2,100}}{S_{a,a+2,100}}$? Can you explain this?

Level 2

If you generalise by changing the value 100 to n here, does this still work?

Can you come up with a conjecture? Can you prove it?

Level 3

Suppose now you pick instead two positive integers a and d, where $d > 1$. You have that $a + d = b$, and $a + 2d = c$. Does your result still hold for $\dfrac{S_{a,b,n} + S_{b,c,n}}{S_{a,c,n}}$?

Does a need to be a whole number here? How about d?

Level 4

Now take any three whole numbers a, b and c where $1 < a < b < c$ and a, b, and c are *not* three consecutive terms from an arithmetic progression.

Does your result for $\dfrac{S_{a,b,n} + S_{b,c,n}}{S_{a,c,n}}$ still hold? Can you explain this?

The Notes for this risp are on page 226; Home Page

Further Risp 35: The k-matrix

Level 1

Define a 2×2 matrix to be a *k-matrix* for a particular non-negative value of k if it is of the form $\begin{pmatrix} a & b \\ kb & a \end{pmatrix}$ where $a, b > 0$.

What do you notice if you add two k-matrices $\begin{pmatrix} a & b \\ kb & a \end{pmatrix}$ and $\begin{pmatrix} c & d \\ kd & c \end{pmatrix}$? If you multiply them?

Level 2

What are the eigenvalues of the k-matrix $\begin{pmatrix} a & b \\ kb & a \end{pmatrix}$?

Say the eigenvalues of $A = \begin{pmatrix} a & b \\ kb & a \end{pmatrix}$ are λ_1 and λ_2, with $\lambda_1 \leq \lambda_2$.

Say the eigenvalues of $B = \begin{pmatrix} c & d \\ kd & c \end{pmatrix}$ are λ_3 and λ_4, with $\lambda_3 \leq \lambda_4$.

What are the eigenvalues λ_5, λ_6 of $A+B$? What are the eigenvalues λ_7, λ_8 of AB? What do you notice?

Can you make a conjecture? Can you prove it?

Are k-matrices the only matrices with this property? Can you find other matrices that follow this rule?

Level 3

What are the eigenvectors of the k-matrix $\begin{pmatrix} a & b \\ kb & a \end{pmatrix}$?

What do these add to? What is their scalar product? What do you notice?

Level 4

Do the identity matrices for addition and multiplication qualify as k-matrices?

What can you say about the inverse of a k-matrix under addition? Under multiplication?

Does this enable you to say anything further about k-matrices?

The Notes for this risp are on page 229; Home Page

Further Risp 36: The Cycling Lines

This risp assumes you are working in a group of three!

Fact; the two lines

$$\underline{r} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \lambda \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad \text{and} \quad \underline{r} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} + \mu \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

meet if and only if

$$\begin{vmatrix} p_1 & q_1 & t_1 \\ p_2 & q_2 & t_2 \\ p_3 & q_3 & t_3 \end{vmatrix} = \begin{vmatrix} s_1 & q_1 & t_1 \\ s_2 & q_2 & t_2 \\ s_3 & q_3 & t_3 \end{vmatrix}$$

Level 1

Consider the sequences $1, a^2, a, 2, 2a^2, 2a, 3, \ldots 5a, 6, 6a^2, 6a$. You can form from this the three vector lines

$$\underline{r} = \begin{pmatrix} 1 \\ a \\ 2a^2 \end{pmatrix} + \lambda \begin{pmatrix} a^2 \\ 2 \\ 2a \end{pmatrix}, \underline{r} = \begin{pmatrix} 3 \\ 3a \\ 4a^2 \end{pmatrix} + \mu \begin{pmatrix} 3a^2 \\ 4 \\ 4a \end{pmatrix}, \underline{r} = \begin{pmatrix} 5 \\ 5a \\ 6a^2 \end{pmatrix} + \delta \begin{pmatrix} 5a^2 \\ 6 \\ 6a \end{pmatrix}.$$

Each of you now pick a different pair of lines. Using the determinant fact above, can you show that your pair of lines meet? Where do your pair of lines meet?

Level 2

These three meeting points form a triangle that defines a plane; can you find its equation? In how many different ways can you write this equation?

Level 3

Is there any value of a for which these three lines meet at a single point?

Level 4

What happens to the angles in the triangle as a increases?

The Notes for this risp are on page 233; Home Page

Further Risp 37: Complex Loci

Level 1

If p is a complex number, and q is a positive real number, what does the curve $|z - p| = q$ look like in the Argand diagram ?

How about the curve $\arg(z - p) = q$? Can you explain why you get these curves?

Level 2

Pick two non-zero complex numbers, say $u = a + bi$ and $v = c + di$ with $b, d > 0$.

On the Argand diagram, sketch the locus A of a point z travelling so that $|z - u| = |v|$.

Now do the same for a point w travelling along a locus B so that $|w - v| = |u|$.

Level 3

Can you be sure A and B will always cross?

Do A and B ever touch? If they do, is there anything special about your choice of u and v in this case?

If one of A and B goes through the origin, what else can you say? What does this implies about your choice of u and v?

Level 4

Add the curves $\arg(z - u) = \arg(v)$ (call this P) and $\arg(z - v) = \arg(u)$ (call this Q) to your original diagram. What do you notice about the point where P and Q meet?

The Notes for this risp are on page 236; Home Page

Further Risp 38: A Reduction Formula

Level 1

Working in polar coordinates, draw $r = \cos\theta$ and $r = \sin\theta$ on the same diagram. Can you explain why you get these curves?

The two graphs enclose an area in the first quadrant; how big is this area?

Level 2

What happens if you now plot together $r = \cos^n\theta$ and $r = \sin^n\theta$ (where n is a positive integer) using a graphing program, and vary n?

Do the two curves always overlap in the first quadrant as n increases?

Level 3

Let's say the value of any overlap area for some value of n is I_n; can you find I_n for any given n?

Can you reduce I_n to an integral of the same type but involving a smaller value of n?

Level 4

Calculate I_n for n from 1 to 5.

How could you explore the idea that (for some constants a and b) a good approximate model for I_n (for small n) might be $I_n = ae^{bn}$?

The Notes for this risp are on page 239; Home Page

Further Risp 39: Friendly Rational Functions

Level 1

What is $\int \frac{f'(x)}{f(x)}dx$?

Level 2

The polynomial $f(x) = x^3 + 3x^2 + 12x + 10$ has a curious property; let

$$g(x) = \frac{f'(x)}{f(x)}.$$

What is $\int g(x)dx$? What is $\int \frac{1}{g(x)}dx$? What do you notice?

Level 3

If the rational function $g(x)$ is of the form $\frac{f'(x)}{f(x)}$ where $f(x)$ is a polynomial function, say g is *simple* (as in 'simple to integrate').

If $g(x)$ is simple and also $\frac{1}{g(x)} = px + q + h(x)$, where $h(x)$ is simple, then say $f(x)$ is *friendly* (since both $g(x)$ and $\frac{1}{g(x)}$ are easy to integrate).

If $f(x)$ is a quadratic polynomial, when is $f(x)$ friendly?

Level 4

You are told the quartic polynomial $x^4 + 4x^3 + cx^2 + dx + e$ is friendly. Can you find c, d and e?

What differential equation must the degree-n polynomial $f(x)$ satisfy to be friendly?

Level 5

Suppose that $f(x)$ integrates to $g(x) + c$, and that also $\frac{1}{f(x)}$ integrates to $\frac{1}{g(x)} + C$. What could $f(x)$ be?

The Notes for this risp are on page 243; Home Page

Further Risp 40: Adding a Rotation to a Reflection

Level 1

What kind of transformation does the matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ represent? What is its determinant?

What kind of transformation does the matrix $B = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ represent? What is its determinant?

What happens if you add these matrices? What is the determinant of $A + B$? What kind of transformation does the matrix $A + B$ represent if $\theta = \alpha$?

Level 2

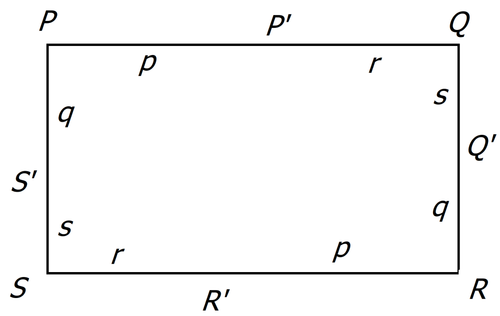
What happens now if you form the matrix $C = aA + bB$, where a and b are both positive constants, and where $0 < \theta, \alpha < \pi/2$ and $0 < \theta + \alpha < \pi/2$.

What's the determinant of this matrix?

Level 3

Suppose $C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and assume that p, q, r and s are all positive.

Now draw this rectangle accurately for your values of p, q, r and s;



Level 4

Can you also draw a parallelogram with diagonals length $2a$ and $2b$ that contain an angle of $\theta + \alpha$?

How is this related to the parallelogram $P'Q'R'S'$?

Level 5

You assumed above that p, q, r and s were all positive; if this is the case, can you show $\theta < \alpha$?

The Notes for this risp are on page 246; Home Page

Further Risps; Reflections

What follows are some random meanderings from me on how these risps might be used; publishing these tasks without any such thoughts feels negligent! In writing about mathematics education there is on the one side the Scylla of stating the blindingly obvious and on the other the Charybdis of being completely impenetrable, so here I try to steer some kind of reasonable middle course; I hope these paragraphs are helpful.

Gatsby Teacher Fellowships

In January 2005, I chanced upon an engaging announcement in a work email that I might well have deleted without any thought. I learnt that the Gatsby Foundation each year offered ten Teacher Fellowships in STEM subjects (Science, Technology, Engineering and Maths), and the next round of awards was fast approaching (sadly the Foundation has since concluded this project). The deal was this:

> *Propose a curriculum experiment for your classroom in the year ahead. If we choose you, we'll support your project with a grant of £3000 for your school or college, and on successful completion, we'll give a further £1000 to you to enliven your personal bank account.*

Application was blessedly simple; a one-page letter, a one-page CV, and one more page itemising the projected spending. My triplet of A4 musings seemed to hit the mark, for a few weeks later I found myself striding towards an interview through the Foundation's impressive lobby near Victoria station. I met with Sue Robinson, representing Gatsby, and with Charlie Stripp, soon to be leading both Mathematics in Education and Industry and the National Centre for Excellence in the Teaching of Mathematics. My pitch received unexpectedly warm attention, and a week later I was whooping with delight to receive the green light. So what exactly had I been taken on to do?

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Risps

The idea behind my Gatsby proposal was in many ways as old as the hills. I offered to create forty open, exploratory mathematical tasks, initially called RSPs (short for 'Rich Starting Points'), which would be true to my key sentence back on page 11. These RSPs would allow my students to investigate in a way that prepared them for any later teacher exposition. I promised to trial these, adding a RSP to my website every week, along with Teacher Notes, across the academic year. I'd build the website myself, which would thus be inevitably a touch primitive, but with its heart hopefully in the right place.

My 'Gatsby Project: Aims, Success Criteria and Milestones' document ended with this;

The RSP project will have been a success first and foremost if students, wherever they are, have a happier and profounder learning experience. How should this outcome be measured? It could be exceedingly hard to put a number on it, but teachers will know if this has been achieved. Their positive feedback on these activities would be a major indicator of success. I know I will pick up from my own students if these RSPs have worked for us.

Gatsby sensibly gave every fellow a mentor to guide them through the pitfalls that every fellowship would doubtless encounter. Did I look especially needy? For I was given two such mentors, Bernard Murphy, from MEI, and Susan Wall, a previous fellow herself. It was Bernard who questioned my description of the letters RSP as 'an acronym'.

'Surely an acronym has to be a word?' he asked. To add the 'i' to 'RSP' was the work of a moment, and so risps were born.

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The Risps Website

My Gatsby Fellowship year fell into a gentle routine. Every week on Wednesday, my day off, I would translate my latest risp into raw html, and upload it onto the website. I made the mistake of launching the site to much fanfare at a premature stage; many of the people I alerted to its existence were bemused to find only a small handful of tasks there. But others saw what I was trying to do, and the emails of support I received were an encouraging surprise.

In college most classes, most days, began for me with a risp, and I took home the feedback I received and built it into improvements. My students knew what was happening as they took to this new way of working, and they were glad to be guinea pigs.

The year flew past, the website arrived on time, Gatsby gave me the thumbs up and my final payment arrived safely. Since then, the feedback has been warm enough for me to be once asked for a selfie at a conference (I know, I know). At one such conference, hosted by the Association of Teachers of Mathematics in 2006, my risp workshop was called 'Would you sow mathematics without ploughing first?' This title probably needs an explanation.

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Ploughing

Teaching is often compared in my mind with farming. There's the rhythm of the agricultural year, that matches the rhythm of the educational year; our annual exam results might equate with some kind of harvest.

But our lessons need to have a rhythm too; ploughing, sowing, growth and reaping. What place does a risp have in the analogy?

No farmer would attempt to sow without ploughing first. A farmer who does such a thing is akin to the teacher who launches straight into unmotivated theory at the start of a lesson, only to experience their students as stony ground.

Using a risp prepares the seed-bed, stirring up the group and inducing a receptivity. Students take a while to get used to this, but when they do, the benefits can be immense.

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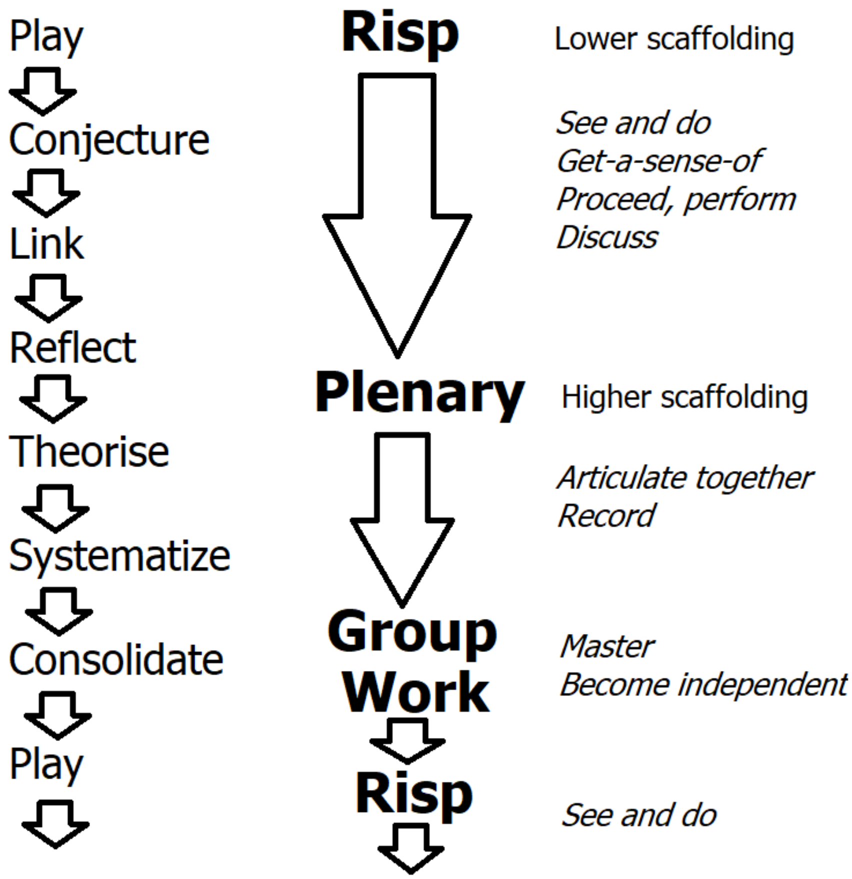
The Risp Cycle



This is how risp work felt to me when it was going well. The key words for me here are ‘play’ and ‘systematize’. Play, because that is often not the notion students have of maths when they arrive onto their A Level course. The idea is to encourage brainstorming with a risp, where being ‘right’ or ‘wrong’ is not the most important thing. This can be a real leveller; ‘weak’ students can play with much more abandon than ‘strong’ students who may not enjoy departing from solid ground. But ‘systematize’; something needs to be distilled from the play. The risp has not been set at random; learning will be in the teacher’s mind, which is not to say unexpected results won’t happen. But there are mathematical structures in student heads that need to be augmented in an organic way. Hopefully the risp will have prepared the ground for the teacher exposition and discussion that helps this along.

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Curiosity

Everything with a rich task hangs on sparking curiosity. We all know what it feels like to see a problem, and to hear the words, 'Hey! I wonder...' pop into our head. I had this sensation last night, as I ran into this question; what proportion of quadratic equations factorise? A whole collection of exciting emotions and thoughts jostled together in my mind. What does this question mean? Can I come up with something for small numbers? Would running a computer program be a good idea? If we say (a, b, c) is an F-triple if $ax^2 + bx + c$ factorises, then are there more or fewer Pythagorean triples than F-triples?

This curiosity, for that's what it is, is pretty much instantaneous with me. It's a hunger that won't go away until I've done something on this problem; it can therefore be annoying in the extreme, as it prevents me paying from attention to other things until I get a break. Where does it come from? It's a mystery. Investigating the problem may bring great pain and frustration in its wake. If I can't make the headway I would like, and so I may even view this capacity for mathematical curiosity as a curse. But then, the satisfaction that comes from resolving my curiosity can be so great that it outweighs everything else.

As I began teaching an A Level class, I often found the levels of mathematical curiosity to be low. For most students, maths was something they were doing because they were good at it, or because they needed it for another subject. Often their previous learning in maths had hindered them in viewing it as beautiful. When asked to name a piece of maths they they'd liked, or had thought was neat, they rarely had an answer. When I suggested to them that an argument was 'elegant', it became a running gag; every time I put a proof on the board, I heard, 'Would you say that was 'elegant', Jonny?'

So cultivating a sense of mathematical curiosity in my students had to be a gradual yet vital undertaking. Everyone likes a magic trick, and even my most cynical students would be impressed if I chose the right mathematical sleight of hand. From there to rich tasks; I knew that if there was no sense of wanting to resolve something, the task would miss its point. There had to be in such a task at least something akin to a magic trick that my students could appreciate. Maybe the proof of anything non-trivial can be seen as such a magic trick.

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Teacher Talk

I think anyone who suggests that an A Level Further Maths course can be delivered without any front-of-the-class teacher exposition is insane. But I think it's entirely reasonable to ask about the quality and quantity of that exposition, which may mean asking, 'What happened to prepare the ground?'

If I go to a conference, there are likely to be range of sessions on offer. Some will be in tune with my key sentence at the start of this book, some won't. My heart sinks when a session turns out to be a straight lecture. I want the chance to think for myself without the pressure of having to do this in public, to chat with my colleagues, to have a possibly inconsequential play with the mathematics. When we're then gathered back together by our host to systematize what we have learnt, I'm curious to know what they and those around me will say. I would say this is true for most students in most mathematics classrooms at whatever level. There may be some cautious souls who would rather follow a lecture, but maybe they should be encouraged to spread their wings rather more.

What would 'rich' teacher exposition look like? For me it required improvisation, and would be enlivened by student contributions. I eventually arrived at what I called *Whole Class Conversation* (WCC).

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Whole Class Conversation

This is taken from a Powerpoint I used to show my classes at the start of the year.

Welcome!

This room looks good! So there are tables ... and we get to choose where we sit... with our mates! This must mean that Jonny wants us to talk to each other! What a cool teacher!

Can I say something?

No, we're busy talking to each other! Only kidding!

What I wanted to say was... indeed, I hope you will talk to each other. I'd like you to learn maths from each other. I hope you learn more maths from each other than you do from me. You discussing maths with each other should be the key activity in this room. But...

There'll be times when you need to put your local discussions to one side, and come together as a whole class. A Level Maths students *always* say that they value clear explanations from the teacher, yet I can't sensibly give the same explanation to fifteen different people one by one.

But on the other hand, I don't want to lecture you (lectures may work at university, but they don't, as a rule, work here in a sixth form college.) So instead, we'll have ... **Whole Class Conversations**.

Rule One; I'm leading the conversation.

Rule Two; contributions are welcome. If you want to contribute, raise your hand, or wait for an appropriate gap in the conversation. I'll make sure I leave gaps for you to say something in.

Rule Three; there should only be one person talking at a time.

Can't I even talk to a mate about the maths? To help him?

Sadly, no. It's amazing how even one tiny little extraneous chat breaks the conversation for everyone else.

Whole Class Conversation requires sensitivity, courtesy, and self-discipline. Its hard. Harder than talking to your mates. You'll need to turn to look at the board; turn to listen to a fellow-student as she says something; bite your tongue when you want to talk to a mate.

I promise that I will not overdo WCC, but... the thing is... mathematics itself is a conversation. One of the most beautiful, powerful, and rewarding conversations that people have ever started. Let's join in...

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Time

One of the commonest things I hear when I try to promote the use of rich tasks is, 'I'd love to, but I haven't got the time.' I have a certain sympathy with this. It's easy to run an A Level Maths course if you don't need to worry about time. We would let a course designed to take 60 classroom hours last for 120, and enjoy open tasks to our heart's content. Unfortunately, the likelihood is that with current funding levels we'll be asked to teach that 60-hour course in 45 hours. So how do we live out our commitment to rich tasks in that scenario?

We'll have to conclude that every second in a maths classroom is precious. We just don't have enough time to do the job as we would like, so we have to be creative. Some people's reaction to having not enough time is to teach defensively, that is, in a way that deflects criticism from management or parents or inspectors or students by being 'safe'. Yet regressing to a more chalk-and-talk model is in fact wasting time if your unengaged students are learning nothing that way. I know to my cost that it's quite possible for an unrich classroom to appear to be functioning beautifully, for the students to listen politely, for the atmosphere to be friendly, for the submitted homework to be apparently good, and yet for the results at the end (at least for the students who have found the course hardest) to be dire. If you instead build a classroom that insists that students get their mathematical hands dirty, that allows teacher exposition only when the ground for that has been prepared, and when students are seeking such exposition to systematise what they've already been discovering, that's when real learning will take place that can carry over into the exam and beyond.

But there is still not enough time, and it's likely there will continue to be not enough time. That's why open tasks need to be carefully written, and carefully chosen. The more directly they practice syllabus skills, the safer it is to set them. Exploring off-the-syllabus material has its place, but time constraints mean this is probably best done outside the lesson. I've done my best here to land at least the start of each Further Risp soundly in the syllabus content. Of course, tasks often develop a life of their own, and can disappear off-piste. There are tasks in this collection that are strictly speaking A Level tasks, but which perhaps require a Further Maths outlook.

I found the best time to set these open tasks was at the start of the lesson, as students drifted in. They became used to the fact that there would be work on the board, and they eventually sat and started without my needing to say anything. Then the improvised exposition, then the engaging consolidation, splitting the lesson into three thirds. There is time for rich tasks this way.

Our college ran on lessons of 90 minutes. An old-style transmission lesson (speaking crudely) might divide this into;

45 minutes of exposition, 45 minutes of practicing examples,

while the risp lesson I was now offering split into

30 minutes of exploration, 30 minutes exposition, 30 minutes of practice via group work.

I argued that the exploration time, on the right task, would motivate and enrich my exposition, which could thus be shortened. The practice at the end could have some underlying motivation too, leading to a more exploratory way to increase procedural fluency.

The new phenomenon of 'the flipped classroom' is emerging that rather stands all this on its head. The deal here is that students tackle the theory for themselves in homework outside the classroom, which frees up time in the classroom for the kind kind of open activities this book contains. Now that there are a wealth of study videos and ebooks available to students online, they can assimilate concepts on their way home on the bus. It remains to be seen where this idea ends up, but it's an exciting prospect.

I spoke to an advisory teacher recently, who said how depressing it was that there was still so much chalk-and-talk maths teaching around these days. I felt like saying, 'But that's a job you've left.' It's amazing how much easier classroom teaching gets when you're not doing it any more. Teaching is tough; the pressures are relentless. And yet my advisory teacher friend is right too; chalk-and-talk is not the most effective thing to be doing, and I would guess it becomes less effective the more our students engage with new technology. Rich and active ways of learning must provide the key modus operandi in our classrooms.

Yet maybe it's possible too to have too much time. As Leonard Bernstein famously once said, 'To achieve great things, two things are needed; a plan, and not quite enough time.'

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The Rule of Thirds

We all teach a range of 'ability' in our classes, but who possesses which ability can vary from lesson to lesson, and in some classes the variation is greater than in others. Ability is a complicated notion that is under the microscope for teachers at the moment, as phrases like 'growth mindset' are more and more common in schools. We're all being encouraged to say that a student's ability is not set in stone, but is something that hard work can transform. I've seen students with weak GCSE results open their wings and fly at A Level, and I've also seen some who scored brilliantly at GCSE peak at that point, and later fade away (maybe to speed up again later, who knows). Perhaps teachers are too willing to judge too early and too finally on what a student can or cannot achieve.

The great thing about working with rich resources is that the task can always be extended further, so a student is not trapped into a lightweight activity that fails to demonstrate what they can do. With a risp, there is always the chance for students to surprise us and challenge our preconceptions. ‘Differentiation’ is no longer the unquestioned good that it used to be, but working with a rich task should keep educationalists of whatever hue happy, as different students find different ways to move forwards.

So this is my Rule of Thirds. Whenever we set a task, there’ll be a variety of outcomes. I came up with this rule for my risps;

The first third of the task should be accessible immediately to everyone.

The second third of the task should be immediately accessible to two-thirds of the class, and moderately accessible to the other third.

The third third should be immediately accessible to a third of the class, and moderately accessible to another third.

Who was in which third could vary, and could be hard to predict. ‘Weaker students’ are often more fearless with risps, while their ‘stronger’ colleagues might be more unwilling to let themselves go mathematically. As I’ve said before, risps can be a leveller.

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Exam Results

It’s possible to detect a sentiment in maths education (I’m not sure how widespread it is, and I’m exaggerating for effect) that exams are rather dirty things that no real teacher ever deigns to think about for long. They are at best a duty, a distraction that disrupts the genuine work of helping students to build in their heads an organic understanding of maths, and one should just pause for a few weeks of disdainful past paper work in early summer before returning to the real task in hand.

This is a caricature, but it's one the rich task teacher may be accused of resembling. There are educational institutions that have reputations as exam factories. Everyone is expected to bow down to end of year grades, and if that means teaching to the test, so be it. Rich tasks might be tolerated if results don't suffer, but if they do, for whatever reason, that teacher may be challenged over their methods.

There are plenty of people working in education who are no longer judged by exams. I remember being at a conference once and being asked, 'How do you approach calculus in your classroom?' I replied that I wanted all my students to cotton on to differentiating as something easy; in fact, I thought differentiating was an area where the students in my class who were finding the course hard could guarantee to pick up marks. My reply went down, well, let's say, badly. Yet is it really off limits to ask where your students are going to pick up easier marks in the exam?

I was always conscious of playing two games in the classroom. The first was the exam game, and the second was infinitely more interesting game that is mathematics. Of course, the second was more important in any ultimate sense, but I had to play the exam game too. Not to do so would have been to short-change my students and my colleagues. I can vividly recall the day the value-added results were handed out to the staff at my college. Some teachers always got fantastic results, and they walked calmly around the gym, quietly superior. Some usually did, but had under-achieved this year, and they could be seen in anxious huddles, wondering what they'd done wrong. Others usually had modest results, but this year they'd entered the stratosphere, and were slumped in a dazed euphoria. Other departments struggled to be average every year, and their faces were their standard stricken selves, depressed by their unrelenting mediocrity despite what may have been astonishing efforts. The amalgamation of these grades represented the most telling advertisement our institution could broadcast. The teachers who 'failed' year after year were whispered about, and relevant Heads of Department received tight-lipped managerial warnings.

But quite apart from this, don't we want our students, those we have worked so hard alongside for so many hours, to achieve their goals, to get to the right university, to feel delighted as they open that envelope? Of course we do, and I have never met a teacher who would say anything different. So yes, we have to play both games, and doing well at both is hopefully not contradictory.

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Problem-solving

Stephane Grappelli, the doyen of jazz violinists, was once asked by an over-earnest acolyte who was writing a book on the master, 'What's the secret of your bow technique?' Apparently Stephane took a sip from his glass of wine, and replied impishly, 'The bow goes up, and the bow comes down.' The author failed to smile before noting the apercu, and it duly appeared in his book.

There's a similar story I like concerning an eminent maths teacher (sadly I can't remember her name) who was asked how she went about teaching the art of problem-solving. She thought for a moment before announcing, 'I set my students problems, and I don't tell them how to do them.'

I'm not sure quite why I like this story; maybe it's because it shows someone refusing to mystify.

Yet is it that obvious? I'm at a Parents' Consultation Evening, and Alice's mother is not happy.

'You put problems on the board, and you don't tell people how to do them,' she says accusingly.

I look at Alice, who is a timid soul, someone who could use a greater belief in her abilities. GCSE maths was a breeze for her, but the rigours of

A Level are taking their toll. She looks at me as if she feels guilty about betraying me to her mother.

I may be at fault here, at least partly. Maybe I'm not choosing the right challenges for Alice. Yet...no one of us knows the world that Alice is heading into, but we can be sure, however, that it'll require her to be more creative than simply copying what her teacher does. Demonstrating a question and then putting up an identical one but with the numbers changed is not going to work as a long-term strategy.

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Mathematical Etudes

As a teenager, I studied classical fiddle for many years. My teacher gave me a green book called 'Shradieck; studies for the violin', that featured strongly in my practice. Each of these testing exercises isolated a particular skill, a specialised fingering, or a bowing style, and offered a 'tune' that would efficiently channel my efforts into learning this new technique.

Henry Shradieck is not a composer that one would hurry to the concert-hall to catch today. A performance of these studies might attract a few hard core violin teachers but few others. His studies are effective (they are still much used) but they don't embody much in the way of artistic merit.

Colin Foster, Professor in Maths Education at the University of Nottingham, questions this. Is it impossible to write etudes that both target certain skills, and yet charm as beautiful pieces in their own right? The example of Chopin is easy to recruit; his etudes are regularly performed publicly by intrepid pianists.

Can we not do the same in the mathematics classroom? We decide that practice of a skill is required. Rather than unimaginatively printing off some worksheet of twenty unrelated examples, can we construct an open task, one that harnesses curiosity, and which practices the desired technique en passant? These risps are offered in that spirit.

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Competition

There is some light-hearted classroom competition that I think is fine. If you've organised people into teams and you're having a quiz, there would something wrong if there wasn't a hint of friendly competition in the air. The maths conference circuit features a quiz on most occasions. But when teaching, there was much in the way of macho verbal fisticuffs that I always felt I could do without.

I remember saying to one boy (and let's fact it, it is usually boys) whose ego was too wrapped up with coming top, 'This is not a competition, James.'

James's reply was immediate; 'No, but if it was, I'd win.'

These risps are built around the idea that students will discuss in groups. This is not always an easy thing to do. What advice did I pass on to my proteges as they tried to work together? You may be working with your friends, you may not. You need to judge when to make a contribution, and maybe the bigger the breakthrough you've made, maybe the more quietly you should announce your success. Have you been daydreaming, and have you lost the thread of the conversation? What can you do to get yourself back into the right stream of thought? Or maybe you're being crowded out of the discussion, and need to prise out an opening to get yourself back into it. What is certainly true is that an over-powering desire to get there before anyone else will damage the conversation, and may be upsetting to your colleagues. Don't hog more of the talk than is your fair share, but don't hide in the shadows for long periods either. Make yourself contribute something if there has been a lull on your part, and your colleagues will pick up if you have missed something. Let other people breathe, and insist on breathing yourself.

The Association of Teachers of Mathematics wisely says, *'Teaching and learning are cooperative activities. Encouraging a questioning approach and giving due attention to the ideas of others are attitudes to be encouraged. Any possibility of intimidating with mathematical expertise is to be avoided.'*

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Leicester City

When I began this book, the nation was reeling from the most miraculous sporting story for centuries. Leicester City, five thousand to one outsiders at the start of the season, won the Premier League 2015-16 by an extraordinary 10 clear points. This story should make every teacher in every classroom pause before consciously or unconsciously writing off anyone in their care. Are there some students in your class where you would offer five thousand to one odds against them coming top? Yet miracles do happen, and when they do, we should learn the lessons, so they can happen more often.

How did Leicester do it? To my mind, it was a mixture of superb common sense, impressive technical nous and a sizeable dash of serendipity. The players clicked with the new manager, Claudio Ranieri, whose organisation across the season was a tactical master-class, moving from expansive counter-attacking at the start to a tighter sequence of 1-0 wins in the second half of the season. Their team had been modestly assembled (at a tenth of the cost of Chelsea's title-winning team a year earlier), but all the signings proved inspired, as key players hit form together (spending large amounts of money never guarantees this).

Ranieri made few changes to his team across the year; he was more consistent in his selections than any other manager in the league. The back room staff clicked too; their mix of standard training with the more creative kind provided just what the team needed. The club was fortunate, in retrospect, to be knocked out of both cups early. Nor were they in Europe; they played significantly fewer games than their rivals, and were able to take an extra day off each week. They played hard, but they rested hard too.

But even now, we've not reached the heart of their success. Ranieri exuded a calmness, a kind of inspired humility, even when those more likely to win the title indulged in the inevitable mind games. The team fought for each other with a relentless esprit de corps. They were everyone's second favorite team, and they rode a wave of goodwill that took them to the title.

It's the teacher's job to try to engineer this kind of outcome with the possibly unpromising classes that face them at the beginning of each year (in the 2014-15 season, Leicester City finished 14th). Their story is the equivalent of every student in our maths group getting an A*. If we all 'did a Ranieri', is that impossible? Leicester City have proved it can be done.

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Computers

Computers are everywhere, and have transformed teaching and learning in the last few decades, in mathematics perhaps more than anywhere else. Strangely, however (to my mind), in exams we only allow students limited computing power; most of the time computer algebra systems are not allowed. I'd be surprised if this continues for much longer. As calculators replaced log tables, so computers will replace calculators. Of course, some exams do allow computers; I think of the more advanced Decision Maths, or the Further Maths Technology option. Some countries (like Denmark and Australia) combine non-computer papers with computer ones. Nor do all mathematicians use computers; Andrew Wiles, for example, does not include them in his armoury for his ground-breaking work. But once you've tasted the different questions that these drudgery-saving programs enable you to investigate, it's hard to go back.

These risps assume that computing power in all its guises is available to students, and I make no apology for this; any Further Maths student should see computing as their friend, and the more techniques they can master, the better. Teachers don't have to book a dedicated computer room to use these tasks; it may well be appropriate to use a teacher computer in a plenary at the start of the lesson, before inviting students to explore for themselves. Phones could soon become extremely useful.

My guess is the Exam Board Canutes will not be able to hold back the tide for much longer. A different kind of assessment will come along where adeptness with the technology will be rewarded. Of course we want to keep a world where a student can differentiate x without resorting to a machine.

But asking a student to do a difficult differentiation that a machine can do completely accurately in less than a second needs to be challenged. I do understand that our generation has had to cope with a computing revolution unlike anything coming before us, but when it comes to our maths exams, I hope future generations will not look back on us too unkindly.

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How can I use VBA in Excel?

You may never have done any computer programming, but when you're searching for solutions to an equation, a program can be of enormous benefit. It really is easy to get started with coding, and once you've started, there is no telling where you might finish. I'm going to assume you have access to the Microsoft Excel program. Lots of people use Excel without knowing that it has the programming language Visual Basic for Applications built into it. I'll show you how it works.

Open a spreadsheet, and go to the Developer tab. If you don't have that showing, try

File
→ Options
→ Customise ribbon
→ Developer Tab

If this doesn't work, you will need to search for help on the internet.

Now click on 'Insert', and in the ActiveX controls section, click on the Command Button (if you hover over the options, it will tell you which is which). Now click somewhere on the spreadsheet, and the command button should appear. Double click on it, and you get a page where you can enter your Visual Basic code.

Private Sub CommandButton1_Click()
Space for code
End Sub

A simple program might say:

$s = 0$
puts the initial value of s to 0.
For a = 1 to 10
starts with a = 1, increasing a by 1 each time we reach 'Next a', and finishing when a gets to 10.
$s = s + a^2$
puts the new value for s equal to the old value for s plus a squared.
Next a
moves on to the next value of a, going back to the For statement.
Cells(1,1) = s
inserts the final value for s into Row 1, Column 1.

Can you say what the program does? The lines in bold are simply comments, and explain what each line does without affecting the program. Type the program (without the comments) into the space provided, then click on 'Design Mode' on the main toolbar to exit that. Now run the program by clicking on your Command Button. This program adds together the first 10 square numbers, so it should end with 385 in Cell A1.

You may not need much more than For-Next loops, the Cells(a, b) notation and the If-then statement to build some handy programs. A nested pair of For-Next loops look like this:

$s = 0$

For a = 1 to 10

For b = 1 to 10

$s = s + a * b$

Next b

Next a

Cells(1,1) = s

For each of the ten values for a, b runs through 1 to 10, which makes 100 calculations of s in total. This program adds together all the numbers in the white area in the grid below. After running the program, the answer 3025 will hopefully be in Cell A1.

x	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Programs can help us with searching for an answer to a problem. I've chosen the Fermat-Catalan conjecture to show this, the result of merging Fermat's Last Theorem with Catalan's Theorem. Fermat's Last Theorem famously says $x^n + y^n = z^n$ has no whole number solutions for x, y and z if n

is greater than 2 (this was proved by Andrew Wiles in 1994). Catalan's Theorem (suggested in 1844) states that $x^a - y^b = 1$ has only one solution in whole numbers, which is $3^2 - 2^3 = 1$ (this was proved by Preda Mihailescu in 2002).

So what might the love child of these two great theorems look like? With almost any conjecture, it's possible to make it more general, and that creates something harder to prove. We're going to try to find solutions to the equation $a^m + b^n = c^k$. How many whole number solutions can we find to this? What happens if one or more of m, n and k is 1? What if m, n and k are all 2? What if two of m, n and k are 2? What if m, n and k are all 3? If we insist that $\frac{1}{m} + \frac{1}{n} + \frac{1}{k} < 1$, then we guarantee that m, n and k are not all small. Can we find a solution for (a, b, c, m, n, k) with this restriction? How could we find other solutions? Does our equation have infinitely many solutions, or finitely many?

If you're going to investigate conjectures, using a computer is common practice. It can save you time by finding you a quick counterexample, and stopping you from disappearing up blind alleys. But mathematicians disagree on how universally helpful they are; we need to beware of leaping onto a computer to resolve something when a little hard thought would answer the question for us immediately. Sometimes a pencil and paper are all we need.

How could we code a program to tackle the Fermat-Catalan conjecture? A program to search for solutions might look like this;

```
s = Cells(1,7)
j=1
For a = 1 to s
For b = a+1 to s+1
For c = 1 to 150
For m = 2 to 10
For n = 2 to 10
For k = 2 to 10
If 1/m + 1/n + 1/k < 1 then
If a^m+b^n=c^k then
Cells(j,1)=a
Cells(j,2)=b
Cells(j,3)=c
Cells(j,4)=m
Cells(j,5)=n
Cells(j,6)=k
j=j+1
End if
End if
Cells(1,8)=a
Next k
Next n
Next m
Next c
Next b
Next a
```

This program employs six nested For-Next loops. Clicking out of Design mode and then clicking the Command Button runs the program. Putting $s = 20$ generates the output below. This at least is the start of the output, which I cut short here (pressing Esc gets you out of the program). You can see that the first four solutions have been found (the highlighted cells).

1	2	3	7	3	2	20	5
1	2	3	8	3	2	CommandBut	
1	2	3	9	3	2		
1	2	3	10	3	2		
2	4	2	4	2	5		
2	4	2	6	3	7		
2	4	2	8	4	9		
2	4	8	8	4	3		
2	4	24	9	3	2		
2	7	3	5	2	4		
2	8	2	6	2	7		
2	8	2	9	3	10		
2	8	4	9	3	5		
2	8	32	9	3	2		
2	16	2	8	2	9		
2	16	8	8	2	3		
2	17	71	7	3	2		
3	6	3	3	3	5		
3	6	45	6	4	2		
3	9	54	7	3	2		
3	11	122	5	4	2		
3	18	3	6	3	8		
3	18	9	6	3	4		
4	8	2	3	2	7		
4	16	2	4	2	9		
4	16	8	10	5	7		
4	16	128	10	5	3		

It takes Excel a while to run this. Every time you add another nested loop, the number of calculations increases exponentially (the idea of cell H1 is to keep track of how far the program has run). Sometimes you need to leave a computer running overnight to arrive at an answer. Once you've experienced the thrill of waking to remember that your machine is waiting for you with an answer, I promise you'll never feel the same way about maths (or computing) again!

Warning; remember to save the changes to any program before running it! You don't want Excel to give up, only for you to then lose your program.

These ten solutions to our general equation given below are the only ones known. Our program found four of these, as far as it was allowed to run (Excel can go up to 16 significant figures; larger numbers than this are going to end in approximate results).

$$1^m+2^3=3^2$$
$$2^5+7^2=3^4$$
$$13^2+7^3=2^9$$
$$2^7+17^3=71^2$$
$$3^5+11^4=122^2$$
$$33^8+1549034^2=15613^3$$
$$1414^3+2213459^2=65^7$$
$$9262^3+15312283^2=113^7$$
$$17^7+76271^3=21063928^2$$
$$43^8+96222^3=30042907^2$$

You'll notice that in each of these ten solutions, one of m, n and k is 2. So we can add to the unsolved problem above Beal's Conjecture, which states that if we have a solution to our general equation, then one of m, n and k has to be 2. Beal is a rich man, and he currently offers a prize of a million US dollars to anyone resolving his conjecture one way or the other. This book could be the wisest investment you've ever made.

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Further Risp 1: The Twizzle: Notes

Topic: matrices
Type of task: consolidation
Preliminary knowledge required: matrix multiplication, determinants, rotation/reflection matrices, transpose
Computing needs: calculator for matrix calculations

Level 1

Suppose we take $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$, so $A^Z = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Now $\det(A) = 5$, while $\det(A^Z) = -5$. Does this work in general? If $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, so $A^Z = \begin{pmatrix} q & s \\ p & r \end{pmatrix}$, and we have $\det(A) = ps - qr$, and $\det(A^Z) = qr - sp = -\det(A)$. So we have that $\det(A) = -\det(A^Z)$ for all 2 by 2 matrices A.

Level 2

With 3×3 matrices, we have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} c & f & i \\ b & e & h \\ a & d & g \end{vmatrix}, \det(A^Z) = -\det(A).$$

This is easily proved using the standard methods for finding the determinants of a 3×3 matrix. We have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + dhc + gbf - ceg - fha - ibd, \text{ while}$$
$$\begin{vmatrix} c & f & i \\ b & e & h \\ a & d & g \end{vmatrix} = ceg + bdi + afh - aei - hdc - gfb.$$

Level 3

The matrix for a rotation through θ about the origin is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, while the matrix for a reflection in the line $y = x\tan\theta$ is $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$.

Suppose we have the rotation matrix $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. If we twizzle, we get

$$\begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} -\cos(\frac{\pi}{2}-\theta) & \sin(\frac{\pi}{2}-\theta) \\ \sin(\frac{\pi}{2}-\theta) & \cos(\frac{\pi}{2}-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\frac{\pi}{2}+\theta) & \sin(\frac{\pi}{2}+\theta) \\ \sin(\frac{\pi}{2}+\theta) & -\cos(\frac{\pi}{2}+\theta) \end{pmatrix}$$

This represents a reflection in the line $y = x\tan(\frac{\pi+2\theta}{4})$.

The matrix $R^{ZZ} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$,

which represents a rotation about the origin of $-\theta$.

Suppose now we have a reflection matrix $F = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$.

If you twizzle, you get a matrix $F^Z = \begin{pmatrix} \sin 2\theta & -\cos 2\theta \\ \cos 2\theta & \sin 2\theta \end{pmatrix}$

$$= \begin{pmatrix} \cos\left(\frac{\pi}{2}-2\theta\right) & -\sin\left(\frac{\pi}{2}-2\theta\right) \\ \sin\left(\frac{\pi}{2}-2\theta\right) & \cos\left(\frac{\pi}{2}-2\theta\right) \end{pmatrix},$$

which is the matrix for a rotation about the origin through $\frac{\pi}{2} - 2\theta$.

If you twizzle F twice, you find

$$F^{ZZ} = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\pi-2\theta) & \sin(\pi-2\theta) \\ \sin(\pi-2\theta) & -\cos(\pi-2\theta) \end{pmatrix}$$

which represents a reflection in the line $y = x\tan\left(\frac{\pi}{2}-\theta\right)$, or $y = x\cot\theta$.

Level 4

To prove $(AB)^T = B^T A^T$ for all square matrices A and B of the same order, we can see

$$(AB)^T_{ij} = (AB)_{ji} = \sum_k a_{jk} b_{ki} = \sum_k b_{ki} a_{jk} = \sum_k (B^T_{ik} A^T_{kj}) = (B^T A^T)_{ij}.$$

We also have that $(AB)^{-1} AB = I \Rightarrow (AB)^{-1} = B^{-1} A^{-1}$.

It's easy to find examples to show that in general, $[AB]^Z \neq B^Z A^Z$.

Suppose A is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and B is the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and suppose also that $[AB]^Z = B^Z A^Z$. This means

$$\begin{pmatrix} aq+bs & cq+ds \\ ap+br & cp+dr \end{pmatrix} - \begin{pmatrix} as+bq & cs+dq \\ ar+bp & cr+dp \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} (a-b)(q-s) & (c-d)(q-s) \\ (a-b)(p-r) & (c-d)(p-r) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and so if $(AB)^Z = B^Z A^Z$ for a general matrix $A, p = r, q = s$. Thus

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ p & q \end{pmatrix}\right]^Z = \begin{pmatrix} p & q \\ p & q \end{pmatrix}^Z \begin{pmatrix} a & b \\ c & d \end{pmatrix}^Z$$

for all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We also have that

$$\left[\begin{pmatrix} p & p \\ r & r \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]^Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^Z \begin{pmatrix} p & p \\ r & r \end{pmatrix}^Z$$

for all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. So if we require that $[AB]^Z = B^Z A^Z$ and $[BA]^Z = A^Z B^Z$ for all 2 by 2 matrices A, then $B = \begin{pmatrix} p & p \\ p & p \end{pmatrix}$ for some p.

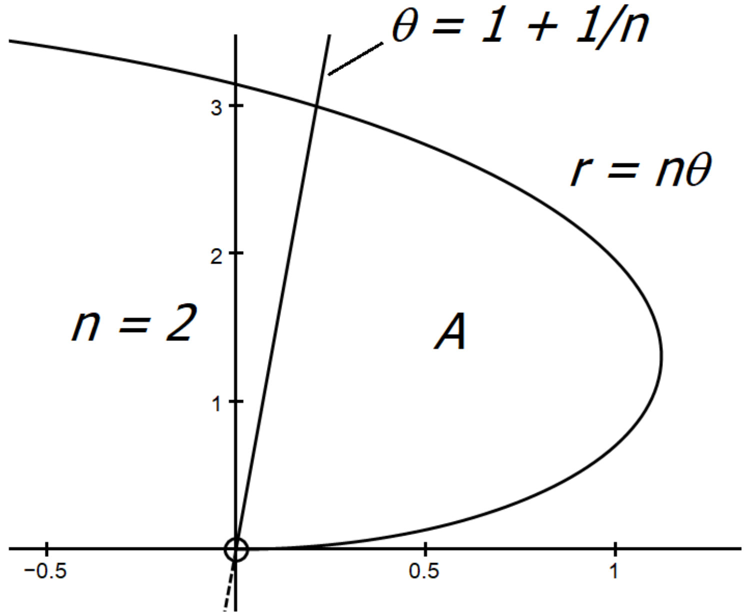
Extra question; can you show the set of matrices $\begin{pmatrix} p & p \\ p & p \end{pmatrix}$ form a group under addition and a group (excluding $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$) under multiplication?

This risp is on page 16; Home Page

Further Risp 2: The Spiral-Line Area: Notes

Topic: polar coordinates
Type of task: consolidation
Preliminary knowledge required: polar area, calculus
Computing needs: a graphing package

Level 1



The curve $r = n\theta$ is a spiral, which 'opens out' as n increases from zero. The curve $\theta = 1 + \dfrac{1}{n}$ is a straight line through the origin which rotates clockwise as n increases.

Level 2

We need the area A in the diagram above. We have

$$\begin{aligned}A &= \int_0^{1+1/n} \frac{1}{2}n^2\theta^2 d\theta \\ &= \left[\frac{n^2\theta^3}{6}\right]_0^{1+1/n} \\ &= \frac{n^2}{6}\left(1+\frac{1}{n}\right)^3 = \frac{(n+1)^3}{6n}.\end{aligned}$$

Differentiating this with respect to n using the quotient rule gives

$$\begin{aligned}\frac{dA}{dn} &= \frac{6n(3(n+1)^2) - 6(n+1)^3}{36n^2} \\ &= \frac{(n+1)^2(2n-1)}{6n^2},\end{aligned}$$

which is 0 when $n = \frac{1}{2}$, where $A = \frac{9}{8}$. We can easily check that this is a minimum point, by drawing the graph of A against n or by differentiating again.

If we take $\theta = a + \frac{b}{n}$ instead, then A has a minimum at $n = \frac{b}{2a}$.

Level 3

What happens if the boundary line is $\theta = e^{-n}$? We have $A = \frac{n^2}{6e^{3n}}$. Differentiating,

$$\frac{dA}{dn} = \frac{ne^{-3n}}{6}(2-3n),$$

which is zero at $n = \frac{2}{3}$. This value for $A \left(\frac{2}{27e^2} \approx 0.01\right)$ is in fact a maximum, as plotting the curve for A against n shows.

Level 4

What boundary line $\theta = f(n)$ will leave the area A constant? We need

$$\frac{n^2}{6}(f(n))^3 = A \Rightarrow f(n) = \sqrt[3]{6A}n^{-2/3} = kn^{-2/3}.$$

As the real cube root of a real number always exists, $f(n)$ exists for all n. It's a good idea to plot $r = n\theta$ and $\theta = n^{-2/3}$ together in a graphing package and observe what happens to A as n varies.

This risp is on page 18; Home Page

Further Risp 3: Can $a + bi = ae^{bi}$? Notes

Topic: complex numbers
Type of task: consolidation
Preliminary knowledge required: Cartesian/polar form for complex numbers
Computing needs: a graphing package

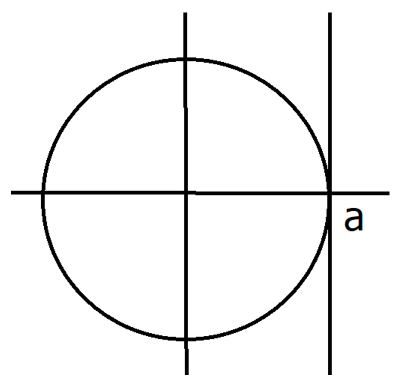
Level 1

The equation $a + bi = ae^{bi}$ yields

$$a = a\cos b, b = a\sin b.$$

We know $a > 0$, so $\cos b = 1 \Rightarrow b = 0$, which is not permitted. So there are no possible values of a and b in these ranges that satisfy this equation.

This diagram helps;



Level 2

The equation $a + bi = 2ae^{bi}$, however, yields

$$a = 2a\cos b, b = 2a\sin b.$$

We know that $a > 0$, so we have $\cos b = \dfrac{1}{2}$, and so $b = \dfrac{\pi}{3}$.

We then have $a = \dfrac{b}{2\sin b} = \dfrac{\pi}{3\sqrt{3}}$.

So there is one solution, $(a, b) = \left(\dfrac{\pi}{3\sqrt{3}}, \dfrac{\pi}{3}\right)$.

What happens with the equation $a + bi = \dfrac{2}{\sqrt{3}} ae^{bi}$? We have

$$a = \frac{2}{\sqrt{3}} a \cos b, b = \frac{2}{\sqrt{3}} a \sin b.$$

We know that $a > 0$, so we have $\cos b = \dfrac{\sqrt{3}}{2}$, and so $b = \dfrac{\pi}{6}$.

We then have $a = \dfrac{\sqrt{3}b}{2\sin b} = \dfrac{\pi\sqrt{3}}{6}$.

Thus our only solution is $(a, b) = \left(\dfrac{\pi\sqrt{3}}{6}, \dfrac{\pi}{6}\right)$.

Level 3

What happens with the general case? We have $a + bi = kae^{bi}$, where k is real and greater than 1. This gives

$$a = ka \cos b, b = ka \sin b.$$

We know that $a > 0$, so we have

$$\cos b = \frac{1}{k} \Rightarrow b = \arccos \frac{1}{k}.$$

Thus $\arccos\left(\dfrac{1}{k}\right) = ka\sqrt{1 - \dfrac{1}{k^2}} \Rightarrow a = \dfrac{\arccos\left(\dfrac{1}{k}\right)}{\sqrt{k^2 - 1}}$.

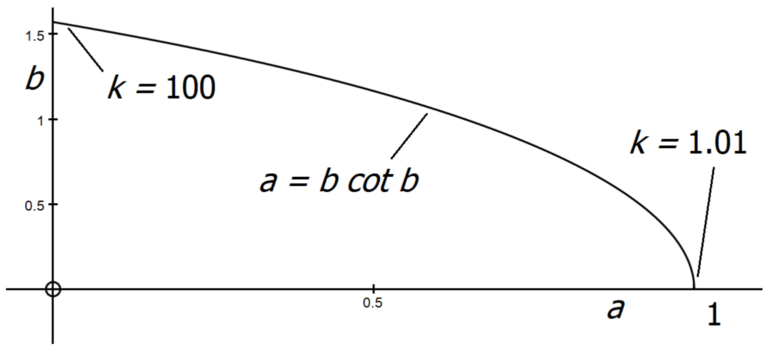
We have one solution; $(a, b) = \left(\dfrac{\arccos\left(\dfrac{1}{k}\right)}{\sqrt{k^2 - 1}}, \arccos\left(\dfrac{1}{k}\right)\right)$.

Level 4

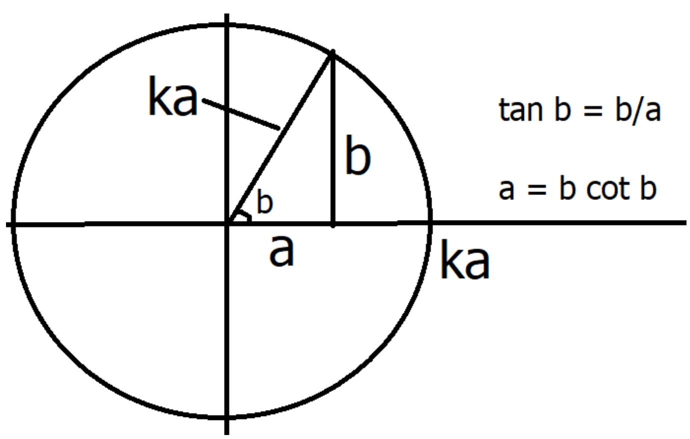
If $b = \arccos\left(\dfrac{1}{k}\right)$ then $k = \sec b$, so we have

$$a = \frac{b}{\sqrt{\sec^2 b - 1}} = \frac{b}{\tan b} = b \cot b.$$

Thus the locus of $a + bi$ for $a > 0, 0 < b < \pi$, is $a = b\cot b$ as k varies between 1 and ∞.



This diagram helps;



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Further Risp 4: Series Arithmagon: Notes

Topic: Maclaurin series
Type of task: introductory
Preliminary knowledge required: differentiation of functions
Computing needs: none

Level 1

For the addition arithmagon, we have $a+b=6, b+c=7, a+c=2$. Subtracting the first two equations gives $a-c=-1$, and adding the third gives $2a=1$. Thus $a=0.5, b=5.5$ and $c=1.5$. This method will always yield exactly one solution, which is real if a, b and c are real.

Level 2

We have that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots,$$

for $|x| < 1$, and

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots,$$

for all x, and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots,$$

also for all x, while

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots,$$

also for all x.

Level 3

So

$$\ln(1+kx) = kx - \frac{k^2x^2}{2} + \frac{k^3x^3}{3} - \frac{k^4x^4}{4} \dots,$$

and

$$ae^{bx} = a + abx + a\frac{b^2x^2}{2!} + a\frac{b^3x^3}{3!} + a\frac{b^4x^4}{4!}\dots.$$

We know that the x^2 coefficients are equal here, so

$$-\frac{k^2}{2} = a\frac{b^2}{2!} \implies k^2 = -ab^2.$$

Now $u\sin(vx) = uvx - u\frac{v^3x^3}{3!} + u\frac{v^5x^5}{5!}\dots$, and we know this shares an x^3-coefficient with $\ln(1+kx)$, thus $-u\frac{v^3}{3!} = \frac{k^3}{3} \implies 2k^3 = -uv^3$.

We also know that $u\sin(vx)$ and ae^{bx} share an x-coefficient, so $uv = ab$.

So we have three equations any solution must satisfy:

$$uv = ab, 2k^3 = -uv^3, k^2 = -ab^2.$$

To find a single viable solution, it seems sensible to put $u = -2 \Rightarrow k = v$. Let's try $v = 4$, say; this gives us $uv = -8 = ab, k^2 = 16 = -ab^2$, and so $b = 2, a = -4$. So this gives one possible solution as $(a, b, k, u, v) = (-4, 2, 4, -2, 4)$.

The expansions for e^{bx} and $\sin(ex)$ are valid for all x, while that for $\ln(1+kx)$ is only valid for $|kx| < 1$, and so $\big|x\big| < \left|\frac{1}{k}\right|$, which in our case becomes $|x| < \frac{1}{4}$.

Level 4

These are our three equations;

$$uv = ab, 2k^3 = -uv^3, k^2 = -ab^2.$$

Let $u = -2p^3$, which gives $k = pv$. Thus $ab^2 = -p^2v^2$, and $ab = -2p^3v$. Dividing these gives $v = 2pb \Rightarrow a = -4p^4$.

Thus one possible two-parameter infinite family of solutions is

$$(a, b, k, u, v) = (-4p^4, b, 2p^2b, -2p^3, 2pb).$$

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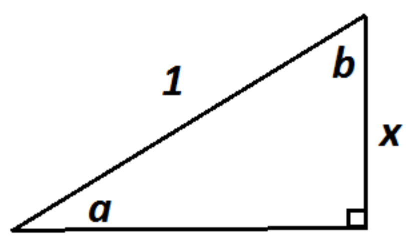
Further Risp 5: Inverse Trig Triangles : Notes

Topic: inverse trig functions
Type of task: introductory
Preliminary knowledge required: none
Computing needs: a graphing package

Level 1

As in the question, we will take angles to be in radians here.

The values $\arcsin x, \arccos x$ and $\arctan x$ can't be the angles of a triangle for any x. It transpires that $\arcsin x + \arccos x$ is constant at $\frac{\pi}{2}$ for all possible x. The diagram below explains why;



In the above we have $a = \arcsin x, b = \arccos x$, and $a + b = \frac{\pi}{2}$.

Thus $\arcsin x + \arccos x + \arctan x = \pi \Rightarrow \arctan x = \frac{\pi}{2} \Rightarrow x = \infty$, which is impossible.

Level 2

We have

$$2 \arcsin x + 2 \arccos x + 2 \arctan x = \pi \Rightarrow \arctan x = 0 \Rightarrow x = 0,$$

which again won't give us a viable triangle. However,

$$1.5 \arcsin x + 1.5 \arccos x + 1.5 \arctan x = \pi \Rightarrow \arctan x = \frac{\pi}{6} \Rightarrow x = \frac{1}{\sqrt{3}},$$

which does work (the angles in radians are 0.9232..., 1.4329..., 0.7854....) If we are given

$$k \arcsin x + k \arccos x + k \arctan x = \pi,$$

then $x = \tan\left(\dfrac{\pi}{k} - \dfrac{\pi}{2}\right)$, which works for $\dfrac{4}{3} < k < 2$, since

$$x < 1 \Rightarrow \frac{\pi}{k} - \frac{\pi}{2} < \frac{\pi}{4} \Rightarrow k > \frac{4}{3}.$$

Level 3

The values $\arctan x, \arctan 2x$ and $\arctan 3x$ CAN be the angles of a triangle. It's useful here to use the formula for adding two arctan expressions;

$$\arctan a + \arctan b = \arctan \frac{a+b}{1-ab}.$$

We can extend this to

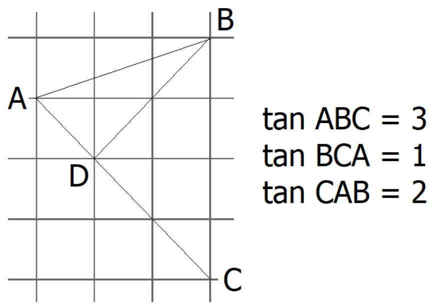
$$\begin{aligned}\arctan a + \arctan b + \arctan c &= \arctan \frac{a+b}{1-ab} + \arctan c \\ &= \arctan \frac{\dfrac{a+b}{1-ab} + c}{1 - \dfrac{a+b}{1-ab}c} \\ &= \arctan \frac{a+b+c-abc}{1-ab-ac-bc}.\end{aligned}$$

Now $\tan \pi = 0$. We need

$$\pi = \arctan x + \arctan 2x + \arctan 3x = \arctan \frac{6x-6x^3}{1-11x^2}.$$

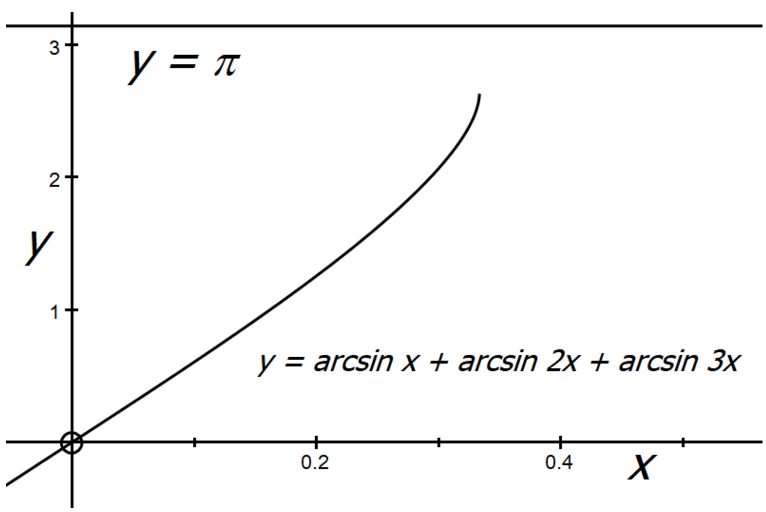
The expression $\dfrac{6x-6x^3}{1-11x^2}$ is zero when $x = -1, 0, 1$, which leads (for our triangle) to $x = 1$.

This diagram shows that $\arctan 1 + \arctan 2 + \arctan 3 = \pi$.



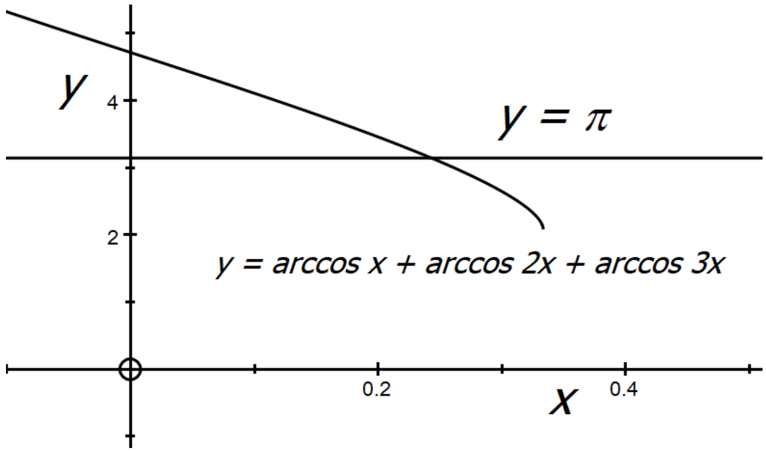
Level 4

Can $\arcsin x$, $\arcsin 2x$ and $\arcsin 3x$ give us the angles of a triangle? This graph tells us they cannot;

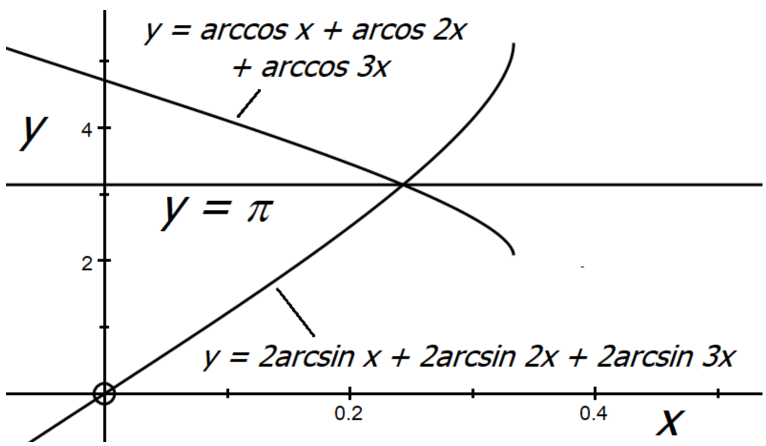


The coordinates of the endpoint here are $\left(\frac{1}{3}, 2.640...\right)$. This can be found by differentiating y and seeing where the gradient is infinite.

Can $\arccos x$, $\arccos 2x$ and $\arccos 3x$ give us the angles of a triangle? This time we have success; by zooming in we have $x = 0.2431(4s.f.)$.



The values $2 \arcsin x, 2 \arcsin 2x$ and $2 \arcsin 3x$ CAN give us the angles of a triangle, but the graph suggests that x here is once again 0.2431....



Can we prove this? We can say

$$
\begin{aligned}
&\pi = 2 \arcsin x + 2 \arcsin(2x) + 2 \arcsin(3x) \\
\Leftrightarrow\ &\pi = 2\left(\frac{\pi}{2} - \arccos x\right) + 2\left(\frac{\pi}{2} - \arccos(2x)\right) + 2\left(\frac{\pi}{2} - \arccos(3x)\right) \\
\Leftrightarrow\ &\pi = 3\pi - 2(\arccos x + \arccos(2x) + \arccos(3x)) \\
\Leftrightarrow\ &\pi = \arccos x + \arccos(2x) + \arccos(3x).
\end{aligned}
$$

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Further Risp 6: FP1 Odd One Out: Notes

Topics: various
Type of task: revision
Preliminary knowledge required: lots
Computing needs: a graphing package

Triplet 1

$4x^3 - 20x^2 + 29x - 28 = 0, 3x^3 - 17x^2 + 33x - 21 = 0$ – the product of their roots is 7, for the other equation it is $\frac{25}{2}$.

$4x^3 - 20x^2 + 29x - 28 = 0, 2x^3 - 10x^2 + 22x - 25 = 0$ – the sum of their roots is 5, for the other equation it is $\frac{17}{3}$.

$3x^3 - 17x^2 + 33x - 21 = 0, 2x^3 - 10x^2 + 22x - 25 = 0$ – the sum of the product of their roots taken in pairs is 11, for the other equation it is $\frac{29}{4}$.

Triplet 2

$$\sum_1^n (5r + 27), \sum_1^n (5r^2 + 2), \sum_1^n (3r^3 + 1).$$

$n = 2$, the second and third sums both add to 29, the first is 69.

$n = 3$, the first and third sums both add to 111, the second is 76.

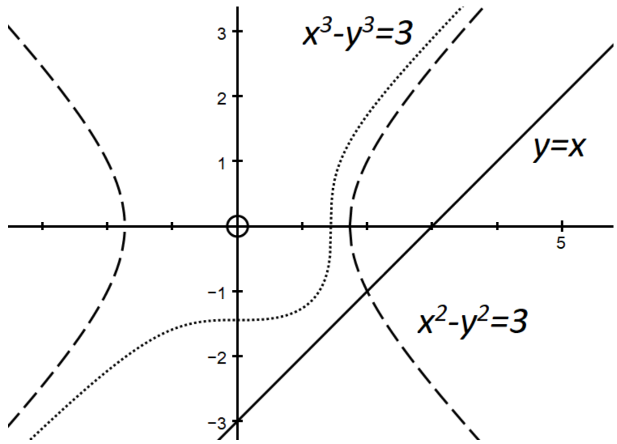
$n = 4$, the first and second sums both add to 158, the third is 304.

Triplet 3

$$x - y = 3 \quad \text{(solid)}, x^2 - y^2 = 3 \quad \text{(dashed)}, x^3 - y^3 = 3 \quad \text{(dotted)}.$$

The first equation gives a straight line, whereas the other two are curves.

The middle equation does not cut the y-axis.



Any polynomial equation in x and y where the degree of each term is less than or equal to two gives a conic (or part of a conic), and any conic has an equation of that kind. The third equation is not a conic (or part of a conic); you cannot cut a cone in such a way as to give this curve as a cross-section.

The middle curve is not symmetrical in $x + y = 0$, whereas the other two are.

The third curve has a stationary point, but the other two curves don't.

The first and second curves intersect, but the third shares no points with the other two.

The second and third curves share an asymptote of $y = x$, while the first curve is parallel to this.

For the first and third curves, $y < x$, while the second curve does see $y > x$ at times.

The first and third curves lie in quadrants 1, 3 and 4, while the second curve lies in all four quadrants.

Triplet 4

$$y = \frac{3x^2+22x+16}{x^2+6x+8}, y = \frac{3x^2+18x+36}{x^2+8x+15}, y = \frac{4x^2+20x+10}{x^2+6x+5}.$$

Curves 2 and 3 have the same vertical asymptote of $x = -5$, while Curve 1 doesn't.

Curves 1 and 3 have a y-intercept of 2, while Curve 2 has a y-intercept of 2.4.

Curves 1 and 2 have a horizontal asymptote of $y = 3$, while the Curve 3 has a horizontal asymptote of $y = 4$.

Triplet 5

$$16e^{i\pi/3}, 2+2\sqrt{3}i, -8+8\sqrt{3}i.$$

The first two complex numbers share the argument of $\frac{\pi}{3}$.

The second and third are both powers of $\sqrt{3}+i = 2e^{i\pi/6}$.

The first and third share a modulus of 16.

Triplet 6

$$e^{2x}\ln(1+5x), e^{5x}, e^{2x}\cos(-3x).$$

If we expand these functions as increasing powers of x, we find that options 2 and 3 have the same constant term, options 1 and 2 have the same x term, while options 1 and 3 have the same x^2 term.

The expansions up to the term in x^2 are respectively as follows:

$$5x - \frac{5x^2}{2}, 1+5x+\frac{25x^2}{2}, 1+2x-\frac{5x^2}{2}.$$

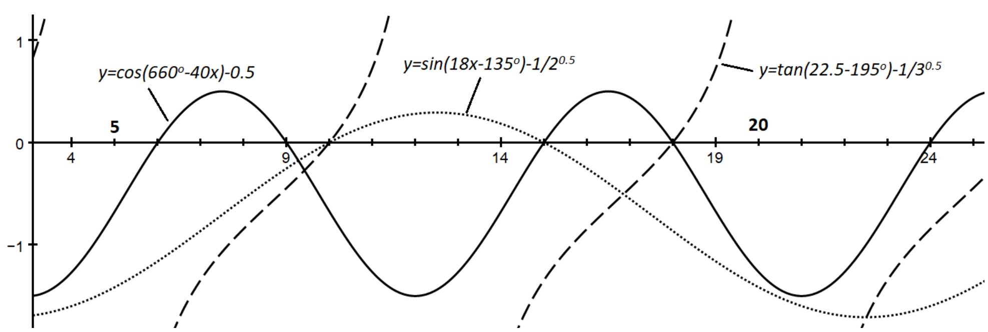
Triplet 7

$$\cos(660^\circ - 40x) = 0.5, \sin(18x - 135^\circ) = \frac{1}{\sqrt{2}}, \tan(22.5x - 195^\circ) = \frac{1}{\sqrt{3}}.$$

For solutions for x in the range 5° to 20°: the sin and cos equations both have the solution 15°, and the sin and tan equations both have the solution 10°, while the cos and tan equations both have the solution 18°.

Here we plot

$$y = \cos(660^\circ - 40x) - 0.5, \quad \text{(solid)}$$
$$y = \sin(18x - 135^\circ) - \frac{1}{\sqrt{2}} \quad \text{(dotted)},$$
$$y = \tan(22.5x - 195^\circ) - \frac{1}{\sqrt{3}} \quad \text{(dashed)}.$$



Triplet 8

Finding the approximate value of a root by

1. interval bisection
2. linear interpolation
3. the Newton-Raphson method.

Methods 1 and 2 rely on examining the change of sign for the function, the Newton-Raphson method does not.

The Newton-Raphson method cannot handle curves that have an infinite gradient at the root ($y = (x^2 - 1)^{\frac{1}{3}}$, for instance) while the other methods can.

Method 1 exactly halves the interval in which the root must lie each time, while the other two methods vary in this regard (they usually do better).

The linear interpolation method can be extended to quadratic interpolation, or cubic interpolation, while the other methods (it could be argued) exist only in a standard version.

Methods 2 and 3 (it could be argued) approximate the curve with a straight line, while 1 does not.

An advanced point; if it takes n steps to achieve our required level of accuracy for the roots, the rate of convergence for bisection is proportional to $\log n$, for linear interpolation it's proportional to n, and for Newton-Raphson it's proportional to n^2 (the number of correct decimal places doubles at each step).

Triplet 9

We wish to use a set of data for x and y to find the values of a and b by plotting a graph. We believe that the law in operation connecting x and y is of the form

$1.y = ab^x, 2.y = ax^b, 3.y = xa^b.$

Law 1 suggests we plot $\log y$ v x, yielding a straight line with gradient $\log b$, and y-intercept $\log a$.

Law 2 suggests we plot $\log y$ v $\log x$, yielding a straight line with gradient b, and y-intercept $\log a$.

Law 3 suggest we plot y v x, yielding a straight line with gradient a^b, and y-intercept 0.

Law 3 doesn't find a and b uniquely, unlike the other two. If this gave us a straight line, this would only tell us a^b.

Law 2 suggests the horizontal axis should represent $\log x$ and not x.

Law 1 gives the gradient as a logarithm, unlike the other two.

Triplet 10

$$\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}, \begin{pmatrix} 3 & -\sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}, \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

Matrix 1 is a rotation through 45° anti-clockwise about the origin combined with an enlargement (of lengths) scale factor 2.

Matrix 2 is is a rotation through 30° combined with an enlargement (of lengths) scale factor $\sqrt{12}$.

Matrix 3 is a reflection in $y = \tan 30°$ anti-clockwise about the origin combined with an enlargement (of lengths) scale factor 2.

So Matrix 2 is the odd one out, since its enlargement scale factor is different (the determinants for matrices 1 and 3 both have size 4).

Matrix 3 is the odd one out since it's a reflection combined with an enlargement, and and not a rotation combined with an enlargement. This means its determinant is negative (the other two have positive determinants).

Matrices 2 and 3 are defined using the angle 30°, while Matrix 1 uses the angle 45°.

The eighth powers of Matrices 1 and 3 are both $\begin{pmatrix} 256 & 0 \\ 0 & 256 \end{pmatrix}$; this is not true for Matrix 2.

This risp is on page 24; Home Page

Further Risp 7: The Which-Clue Matrix: Notes

Topic: matrices, eigenvalues, eigenvectors
Type of task: consolidation
Preliminary knowledge required: matrices, eigenvalues, eigenvectors
Computing needs: a calculator for computing matrix inverses and multiplying matrices

Level 3

If A, B and D are true then since $\lambda_1 + \lambda_2 + \lambda_3 = \text{tr}(X)$, the eigenvalues of X are 8, -5 and -3.

But if C is also true, we know that $\lambda_1\lambda_2\lambda_3 = \det(X) = -160$.

But $8 \times -5 \times -3 = 120 \neq -160$, and so A, B, C and D cannot all be true together.

What about the clues E and F? The eigenvectors of $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ are of the form $\begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ q \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$, so the clues E and F contradict each other.

Level 4

As above, if A, B and D are true then since $\lambda_1 + \lambda_2 + \lambda_3 = \text{tr}(X)$, and the eigenvalues of X are 8, -5 and -3.

If A, B and C are true, then since $\lambda_1\lambda_2\lambda_3 = \det(X) = -160$,

the eigenvalues of X are 8, -5 and 4.

If A, C and D are true, then since

$$8 \times \lambda_2\lambda_3 = \det(X) = -160 \Rightarrow \lambda_2\lambda_3 = -20.$$

We also have that $8+\lambda_2+\lambda_3 = \mathrm{tr}(X) = 0$. Solving these two equations together gives $\lambda_2 = 2, \lambda_3 = -10$.

If B, C and D are true, then since

$$-5 \times \lambda_2\lambda_3 = \det(X) = -160 \Rightarrow \lambda_2\lambda_3 = 32.$$

We also have that $-5+\lambda_2+\lambda_3 = \mathrm{tr}(X) = 0$. However, solving these two equations together gives λ_2, λ_3 to be complex, which is not permitted.

So there are three possibilities for the eigenvalues of X; $(-5,4,8), (-5,-3,8)$ and $(-10,2,8)$. If E is true, then our task is simple, since the eigenvalues of $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ are a, b and c. Thus

$$X = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}, \begin{pmatrix} -5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 8 \end{pmatrix} \text{ or } \begin{pmatrix} -10 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix},$$

and for all of these matrices G is true. If instead F is true, then

$$X \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$\Rightarrow X = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}^{-1}$$

We can now substitute in for λ_1, λ_2 and λ_3. If $3X$ is to have integer coefficients, then it transpires that only one of these matrices will fit the bill; we must have

$$X = \frac{1}{3} \begin{pmatrix} 37 & -17 & -11 \\ 9 & -1 & -9 \\ 44 & -16 & -34 \end{pmatrix}.$$

So we have in total four possibilities for X.

This risp is on page 26; Home Page

Further Risp 8: Induction Number Theory: Notes

Topic: proof by induction
Type of task: introductory
Preliminary knowledge required: language of proof
Computing needs: simple programming is helpful (possibly using Excel); see page 103

Level 1

We aim to show $6|n^3 - n$ for all $n \geq 0$. Let's call the statement '$6|n^3 - n$' S_n.

S_0 states that $6|0$, which is certainly true. Now we assume S_n for some n, so we have $6|n^3 - n$.

Consider $(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1 = (n^3 - n) + 3n(n+1)$.

By our assumption we know $6|n^3 - n$, and since $n(n+1)$ is even, $6|3n(n+1)$ and so $6|(n+1)^3 - (n+1) \Rightarrow S_{n+1}$ is true.

So $S_1 \Rightarrow S_2 \Rightarrow S_3, \ldots$, and by the principle of mathematical induction, we have our result.

Level 2

We aim to show $4^n + 7^n + 1$ is always divisible by 6 for all $n \geq 1$.

As above, let's call the statement '$6|4^n + 7^n + 1$' S_n.

S_1 states that $6|4 + 7 + 1 = 12$, which is certainly true.

Now we assume S_n, so we have $6k = 4^n + 7^n + 1$ for some k. Consider $4^{n+1} + 7^{n+1} + 1 = 4(4^n + 7^n + 1) + 3(7^n) - 3 = 24k + 3(7^n - 1)$.

Since $7^n - 1$ is even, 6 divides both $24k$ and $3(7^n - 1)$, and so S_{n+1} is true.

So $S_1 \Rightarrow S_2 \Rightarrow S_3, \ldots$, and by the principle of mathematical induction, we have our result.

Level 3

We are looking for results of the form $d|a^n + b^n + c$ for all positive n, where $a \leq b < d$, and where $c > 0$.

We are free to say $a \leq b$ without loss of generality, and if $d < a$ or $d < b$, we can just reduce a or b by a multiple of d to give an equivalent d-conjecture where $a \leq b < d$ (we won't want a or b to be multiple of d).

So the Visual Basic program we might write could look like the one one the next page (notes on the program are in bold). The spreadsheet that includes this program is available online at

www.further-risps.co.uk

```
Private Sub CommandButton1_Click()
k = 1
```

k is our row number for printout

```
For a = 2 To 10
```

Values of a will run from 2 to 10

```
For b = a To 10
```

Since a <= b, b starts at a

```
For c = 1 To 10
For d = b + 1 To 10
s = 0
```

s will supply us with a count

```
For n = 1 To 10
x = a ^ n + b ^ n + c
```

x is the main value we are interested in

```
y = x - d * Int(x / d)
```

int(p) means 'the integer part of p'

If d|x then y will be zero.

```
If y = 0 Then
s = s + 1
```

If d|x, then the count s goes up by 1

```
End If
Next n
```

Try the next value for n

```
If s > 6 Then
Cells(k, 1) = a
Cells(k, 2) = b
Cells(k, 3) = c
Cells(k, 4) = d
```

If s gets above 6, this is likely to be a result for us, so we print out a, b, c, d

```
k = k + 1
```

Increase the row number for next time

```
End If
Next d
Next c
Next b
Next a
```

Try out all the other possible values for a, b, c, d less than or equal to 10

```
End Sub
```

For help with using this part of the Excel package, see page 103. If we run the program, part of the output is this:

a	b	c	d
2	2	4	4
2	2	8	4
2	2	8	8
2	4	1	7
2	4	8	7
2	4	8	8
2	6	8	8
3	3	2	4
3	3	6	4
3	3	6	6
3	3	9	9
3	3	10	4
3	4	5	6
3	6	9	9
3	7	6	8
4	4	4	6
4	4	8	8
4	4	10	6
4	6	8	8
4	7	7	9
5	5	6	8
5	5	10	10
5	6	9	10
6	6	8	8
6	6	8	10
6	6	9	9

These results need to be sifted; there are some trivial combinations here. We should add that the computer printout does not prove anything! We will have to supply the logical arguments that do that. But we now have a number of d-conjectures that we might try to prove by induction.

Level 4 and Level 5

Mathematical induction is a powerful tool used by mathematicians to prove things all the time. But it can sometimes be a sledgehammer to crack a nut. Before embarking upon induction, you should try to think of a quicker, more elegant way to accomplish your proof (although of course, if an exam question insists you use induction, that is what you must do).

Can we prove all of the results now on our list like this? We might try, for example, to prove that $8|3^n + 7^n + 6$ (let's call this statement R_n).

We could successfully use exactly the method we used above (the direct approach); we can use the fact that $3^{n+1}+7^{n+1}+6 = 3(3^n+7^n+6)+4(7^n-3)$, where $7^n - 3$ must be even.

Compare that with this less direct approach, that looks at R_{n+2} rather than R_{n+1}. We have

$$8|3^{n+2} + 7^{n+2} + 6 \iff 8|9 \times 3^n + 49 \times 7^n + 6 \iff 8|3^n + 7^n + 6.$$

So $R_n \Rightarrow R_{n+2}$. This time we need both R_1 and R_2 to be true, whereupon $R_1 \Rightarrow R_3 \Rightarrow R_5 \ldots$, and $R_2 \Rightarrow R_4 \Rightarrow R_6 \ldots$, and so R_n is true for all positive n.

We can stretch this idea further. Take T_n to be $9|4^n + 7^n + 7$. The direct approach is possible here, but it's tricky. Let's instead check T_1, T_2 and T_3. We have that T_1 says $9|18$ (true), while T_2 says $9|72$ (true), and T_3 says $9|414$ (true). Now let's assume T_n is true, and now look at T_{n+3}. This says

$$\begin{aligned} &9|4^{n+3} + 7^{n+3} + 7 \\ \iff &9|64 \times 4^n + 343 \times 7^n + 7 \\ \iff &9|4^n + 7^n + 7. \end{aligned}$$

So this time we establish the truth of T_1, T_2, and T_3, and then firstly $T_1 \Rightarrow T_4 \Rightarrow T_7 \ldots$, secondly $T_2 \Rightarrow T_5 \Rightarrow T_8 \ldots$, and thirdly $T_3 \Rightarrow T_6 \Rightarrow T_9 \ldots$, which means T_n is true for all positive n.

When using induction, there are times when this ‘working in leaps’ method is the only one that will give you what you want. You might, for example, try the following question:

$$(\sqrt{2}-1)^2 = \sqrt{9}-\sqrt{8},$$
$$(\sqrt{2}-1)^3 = \sqrt{50}-\sqrt{49}.$$
$$(\sqrt{2}-1)^4 = \sqrt{289}-\sqrt{288}.$$

Is it true that every positive power of $\sqrt{2}-1$ is the difference of the square roots of consecutive integers?

With thanks to MEI for this Item of the month from March 2010*.*

This risp is on page 28; Home Page

Further Risp 9: The Unexpected Group: Notes

Topic: groups, binary operations
Type of task: revision
Preliminary knowledge required:
number sets, binary operations, the group definition
Computing needs: none

Level 1

A group is a set together with a binary operation o so that if a and b are in S, aob is also in S (we have *closure*). There also exists an element e in S (the *identity* element) so that $ae = ea = a$ for all a in S, and for every a in S there exists an element a^{-1} in S (the *inverse* of a) so that $aa^{-1} = a^{-1}a = e$. Additionally we have *associativity.*

$(\mathbb{N}, +), (\mathbb{N}, \times), (\mathbb{Z}, \times)$ are not groups; inverses are a problem. But

$$(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +), (\mathbb{Q} \quad \text{excluding} \quad 0, \times), (\mathbb{R} \quad \text{excluding} \quad 0, \times)$$

are all groups.

Level 2

Do the natural numbers together with $\times$ form a group? Two natural numbers multiply to another, so we have closure, and 1 is an identity. We will need to exclude 0, since that won't have an inverse. Associativity is fine, but inverses in general are an insurmountable problem. We say the natural numbers with $\times$ form a *semi-group.*

Do the integers together with $\times$ form a group? We have an identity element in 1, we have closure in that every pair of integers multiplies to an integer, and this multiplication is associative. But... again, we don't have inverses, so we don't have a group. The integers with $\times$ form a semi-group, but the integers are a group with respect to addition, so overall they form what is called a *ring.*

The rationals together with $\times$ is a group, as long as we exclude zero (we have inverses now). Since no two non-zero rationals multiply to give zero, we still have closure. The story is the same with the reals; we need to exclude zero to form an abelian group here with $\times$. Both the rationals and the reals are abelian groups with respect to addition, and multiplication is distributive over addition; we say in each case they form a *field*.

Level 3

Now in the second diagram we have our 'rectangle' at a tilt of 45 degrees. It's not hard to see that the number of dots coloured for an n by m 'rectangle' is

$$mn + (m-1)(m-1) = 2mn - m - n + 1.$$

So $m \triangle n = 2mn - m - n + 1$. Does working with the natural numbers, the integers, the rationals or the reals give us a group using this as a binary operation?

Do we have an identity element e? That would give $2me - m - e + 1 = m$, or $(2m-1)(e-1) = 0$. This must work for all m, so $e = 1$ is our identity element.

Do we have inverses? This would mean $2mn - m - n + 1 = 1$ if n is the inverse of m, which gives $n = \dfrac{m}{2m-1}$. If m is a natural number other than 1 or 0, this will not be a natural number or integer, so we don't have inverses for those sets. But as long as $m \neq \dfrac{1}{2}$, we do have an inverse in the rationals or the reals.

Let's check associativity. We have

$$\begin{aligned}(m \triangle n) \triangle p &= 2(2mn - m - n + 1)p - (2mn - m - n + 1) - p + 1 \\ &= 4mnp - 2mn - 2mp - 2np + m + n + p.\end{aligned}$$

There's symmetry here. We now expect

$$2m(2np-n-p+1)-m-(2np-n-p+1)+1 = 4mnp-2mn-2mp-2np+m+n+p$$

also, which is true. So (maybe surprisingly) we have associativity; it only remains to check closure. If we take two rational numbers that are not $\dfrac{1}{2}$, can we ever get $\frac{1}{2}$ using this operation? That would mean

$$2mn-m-n+1=\frac{1}{2} \implies 4mn-2m-2n+1=0 \implies (2m-1)(2n-1)=0.$$

So if $m \bigtriangleup n=\frac{1}{2}$, either $m=\frac{1}{2}$, or $n=\frac{1}{2}$. Thus defining multiplication in this way does give a group with the rationals excluding $\frac{1}{2}$, or the reals excluding $\frac{1}{2}$. Clearly from our definition of $\bigtriangleup$, this operation is commutative, and our groups are abelian.

Level 4

We've come some way from the original interpretation of grid-points in a rectangle, but the operation that that suggested has proved to be a helpful one. Turning to the next diagram, we have a group again here, with the set in question being the rationals excluding 0.9 or the reals excluding 0.9.

We can generalise for the rectangle with one pair of sides being at a gradient of s, where s is a positive integer. The operation here is

$$m \bigtriangledown n = mn + (m-1)(n-1)s^2.$$

The identity, as before, is 1. The inverse of $m, m^{-1} = \frac{1+(m-1)s^2}{m+(m-1)s^2}$, as long as $m \neq \frac{s^2}{1+s^2}$. So we need to exclude $\frac{s^2}{1+s^2}$ from our set; will we still have closure? Suppose

$$mn + (m-1)(n-1)s^2 = \frac{s^2}{1+s^2}.$$

This implies (miraculously!) on multiplying out and rearranging, that $(m+ms^2-s^2)(n+ns^2-s^2)=0$. This means that

$$mn + (m-1)(n-1)s^2 = \frac{s^2}{1+s^2} \implies m = \frac{s^2}{1+s^2} \quad \text{OR} \quad n = \frac{s^2}{1+s^2},$$

neither of which can happen, so yes, we have closure if our set is the reals or the rationals excluding $\frac{s^2}{1+s^2}$.

We could go further here, to consider, say, a rectangle with a pair of sides with gradient $\dfrac{p}{q}$, where p and q are primes.

It could be that there is some underlying reason why the group structure is in evidence here, and any such reason will be gratefully received!

This risp is on page 30; Home Page

Further Risp 10: First Order Differential Equations: Notes

Topic: differential equations
Type of task: introductory
Preliminary knowledge required:
variables separable method, integrating factor method
Computing needs: none

Level 1

Given $\dfrac{dy}{dx}+Py=Q$, the integrating factor method suggests that we multiply by $R=e^{\int P\,dx}$.

This means that our equation becomes

$$\frac{d}{dx}Ry = QR \Rightarrow Ry = \int QR\,dx \Rightarrow y = \frac{1}{R}\int QR\,dx.$$

So we have a solution as long as we can integrate both P and QR, which is not guaranteed.

Level 2

The differential equation $\dfrac{dy}{dx}+\dfrac{1}{1+x}y = x^2$ is easy to solve using the integrating factor method (we just multiply by 1+x).

The differential equation $\dfrac{dy}{dx}+\dfrac{1}{\sin x\cos x}y = x\cos x$ is trickier but possible to solve using the integrating factor method. We need to integrate $\dfrac{1}{\sin x\cos x}$, which is $\ln(\tan x)$, and so R is $\tan x$. Multiplying by this, the right hand side becomes $x\sin x$, which we can integrate using integration by parts.

The differential equation $y'+e^{\sqrt{\tan x}}y = e^{\sqrt{\cos x}}$ is impossible to solve using the integrating factor method. We would have to resort to approximate techniques here.

Level 3

So what if $P = Q$? Let's choose $P = Q = x^3 \Rightarrow R = e^{\int x^3\,dx} = e^{x^4/4}$.

The constant of integration does not matter at this point; it disappears in the equation $Ry' + RPy = RQ$. Can you show this?

So we have from $y = \dfrac{1}{R}\int QR\,dx$, that

$$y = e^{-x^4/4}\int x^3 e^{x^4/4}dx = e^{-x^4/4}(e^{x^4/4} + c) = 1 + ce^{-x^4/4}.$$

This seems to work out neatly, and we can try other simple functions that work out neatly too. What if we try the general case?

We have $y = \dfrac{1}{R}\int QR\,dx = \dfrac{1}{e^{\int P dx}}\int Pe^{\int P dx}\,dx$.

What is $\int Pe^{\int P dx}\,dx$? Let $u = \int P\,dx$, so $du = P\,dx$. Then

$$\int Pe^{\int P\,dx}dx = \int e^u\,du = e^u + c = e^{\int P\,dx} + c = R + c.$$

Thus $y = 1 + \dfrac{c}{R} = 1 + ce^{-\int P\,dx}$.

So if we can integrate P, we have a solution to our differential equation.

Level 4

But here is an easier way to solve $\dfrac{dy}{dx} + Py = P$, by separating the variables. We have

$$\frac{dy}{dx} = P(1-y) \Rightarrow \int \frac{1}{1-y}dy = \int Pdx \Rightarrow -\ln(1-y) = \int Pdx,$$

and so

$$y = 1 + ce^{-\int P\,dx},$$

as before.

This risp is on page 32; Home Page

Further Risp 11: Characteristic Equations: Notes

Topic: matrices
Type of task: consolidation
Preliminary knowledge required:
eigenvectors, eigenvalues, characteristic equations
Computing needs: none

Level 1

Suppose that the matrix $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ has the characteristic equation t^2+3t-4.

Thus $t^2+3t-4=0$ is the same as $\det\begin{pmatrix} a-t & b \\ 1 & -t \end{pmatrix}=0$, which gives us $t^2-at-b=0$. So if we choose a = -3, b = 4, everything ties together. There is only one possible choice for a and b.

Level 2

We start with the matrix $\begin{pmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Putting $\det\begin{pmatrix} a-t & b & c \\ 1 & -t & 0 \\ 0 & 1 & -t \end{pmatrix}=0$ gives us the characteristic equation $t^3-at^2-bt-c=0$.

So given the equation t^3+pt^2+qt+r, we can see that the matrix $\begin{pmatrix} -p & -q & -r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ will have this as its characteristic equation.

Level 3

The simplest matrix with eigenvalues α, β and γ is $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$.

What if we insist that the matrix should be of the form $\begin{pmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$?

The characteristic equation can be written as $(t-\alpha)(t-\beta)(t-\gamma)=0$, which is $t^3-(\alpha+\beta+\gamma)t^2+(\alpha\beta+\beta\gamma+\gamma\alpha)t-\alpha\beta\gamma=0$.

On comparing this with t^3-at^2-bt-c, we find

$$a=\alpha+\beta+\gamma, b=-(\alpha\beta+\beta\gamma+\gamma\alpha), c=\alpha\beta\gamma.$$

So the matrix we seek is $\begin{pmatrix} \alpha+\beta+\gamma & -(\alpha\beta+\beta\gamma+\gamma\alpha) & \alpha\beta\gamma \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Level 4

We know $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ has the characteristic equation $t^2-at-b=0$.

What are the eigenvectors? We seek $\begin{pmatrix} x \\ y \end{pmatrix}$ such that

$$\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=t\begin{pmatrix} x \\ y \end{pmatrix}.$$

Using the bottom row, we have $x=ty$, and so the eigenvector for the eigenvalue t is $\begin{pmatrix} t \\ 1 \end{pmatrix}$. We can check this using the top row; $ax+by=tx$ gives the eigenvector $\begin{pmatrix} b \\ t-a \end{pmatrix}$, which is in the direction of $\begin{pmatrix} t \\ 1 \end{pmatrix}$ if and only if $\dfrac{b}{t-a}=\dfrac{t}{1}$, which is true if and only if $t^2-at-b=0$, which we know is true.

In an exactly similar way for 3×3 matrices, we have that the eigenvector of eigenvalue t is $\begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix}$. Notice that if we know the eigenvalues, we can find the eigenvectors without knowing a, b and c.

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Further Risp 12: Sketching a Rational Function: Notes

Topic: graph sketching
Type of task: introductory/revision/consolidation
Preliminary knowledge required: asymptotes, quotient rule
Computing needs: a graphing package

Suppose we pick the numbers $3, 7$, and 9, and say that y has a vertical asymptote at $x = 7$, a horizontal asymptote at $y = 9$, and an x-intercept of 3. We see immediately that as $x \to \infty, y \to a$, so $a = 9$.

Also we have that $x - 7$ divides exactly into $x^2 + x + d$, so $49 + 7 + d = 0$, by the factor theorem, and $d = -56$.

The point $(3, 0)$ is on the curve, so $9a + 3b + c = 0$, which yields

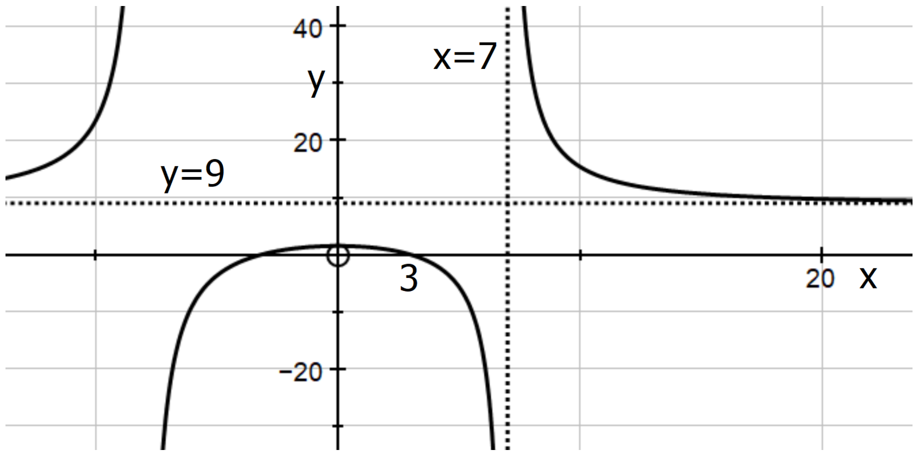
$$c = -81 - 3b.$$

If $y = \dfrac{ax^2 + bx + c}{x^2 + x + d}$, then $y' = \dfrac{(x^2 + x + d)(2ax + b) - (2x + 1)(ax^2 + bx + c)}{(x^2 + x + d)^2}$.

This must equal zero when $x = 0$, so $db = c$.

Substituting in our earlier answers, $-56b = -81 - 3b$, which yields $b = \dfrac{81}{53} = 1.528$, and so $c = -85.58$.

This gives us the following graph:



What happens if we tackle this problem more generally? Suppose we pick the numbers p, q, and r, and say that y has a vertical asymptote at $x = p$, a horizontal asymptote at $y = q$, and an x-intercept of r.

We see immediately that $a = q$.

We know $x - p$ divides exactly into $x^2 + x + d$, so $p^2 + p + d = 0$, by the factor theorem, and $d = -p^2 - p$.

The point $(r, 0)$ is on the curve, so $ar^2 + br + c = 0$, which yields $c = -qr^2 - rb$.

We know $db = c = -qr^2 - rb$, so

$$b = \frac{qr^2}{p^2 + p - r}.$$

So we have a unique answer for (a, b, c, d) unless $p^2 + p = r$, when b (and therefore c) is not defined.

The only time this happens in the given ranges for a, b, c and d is when $p = 1$ and $r = 2$, or $p = 2$ and $r = 6$.

If you picked these values for p and r, you were unfortunate!

Finally, if $d = 0$ then the denominator is 0, which means the curve has a vertical asymptote at $x = 0$, and thus cannot have a turning point there.

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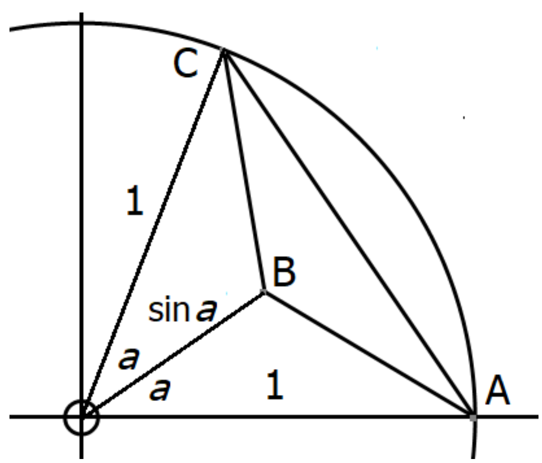
Further Risp 13: The Rotating Triangle: Notes

Topic: polar coordinates
Type of task: introductory
Preliminary knowledge required: none
Computing needs: graphing package essential

Level 1

Our triangle has vertices (in Cartesian coordinates) at

$$A = (1, 0), B = (\sin a \cos a, \sin^2 a), C = (\cos(2a), \sin(2a)).$$



A and C are on the unit circle as a varies.

If $\dfrac{da}{dt} = k$ where k is constant, then A's speed is 0 at time t, B's speed is $\sqrt{\left(\dfrac{dr}{dt}\right)^2 + r^2\left(\dfrac{da}{dt}\right)^2} = \sqrt{k^2\cos^2 a + k^2 \sin^2 a} = k$, and C's speed is $2k$.

Level 2

The triangle is clearly always isosceles, and sometimes equilateral. It can be right-angled, and it can become a horizontal line segment.

The origin is outside the triangle for $a = n\pi$ to $n\pi + \dfrac{\pi}{2}$ where n is integer, and then inside for $a = n\pi + \dfrac{\pi}{2}$ to $(n+1)\pi$.

The locus of B is $r = \sin\theta$, which is a circle.

Level 3

Using the fact that the area of a triangle is $\dfrac{1}{2}ab\sin C$, we have A, the area of our triangle, is

$$A = \frac{1}{2} \times 1 \times 1 \times \sin(2a) - 2\left(\frac{1}{2} \times \sin a \times \sin a\right)$$

$$= \frac{\sin(2a)}{2} - \sin^2 a.$$

We should add that A is in fact the absolute value of this, since an area is always positive. Differentiating,

$$\frac{dA}{da} = \cos(2a) - 2\sin a\cos a = \cos(2a) - \sin(2a),$$

which is 0 when $\tan(2a) = 1 \Rightarrow 2a = \dfrac{\pi}{4} + n\pi \Rightarrow a = \dfrac{\pi}{8} + \dfrac{n\pi}{2}$.

This gives us a maximum for A of 1.207... when $a = \dfrac{5\pi}{8}$.

This maximum looks as though it might occur when the triangle is equilateral, but investigating the angles and sides shows this is not the case.

When is the triangle equilateral? We need (using the cosine rule)

$$\sqrt{1 + \sin^2 a - 2\sin a\cos a} = \sqrt{1 + 1 - 2\cos(2a)}$$

$$\Rightarrow 1 + \sin^2 a - 2\sin a\cos a = 2 - 2\cos(2a)$$

$$\Rightarrow 1 + \frac{1 - \cos(2a)}{2} - \sin(2a) = 2 - 2\cos(2a)$$

$$\Rightarrow 3\cos(2a) - 2\sin(2a) = 1$$

$$\Rightarrow \cos(2a + \arctan\frac{2}{3}) = \frac{1}{\sqrt{13}}$$

$$\Rightarrow a = 0.3508... + n\pi, 2.2028... + n\pi.$$

The value of $|A|$ for $a = 2.2028$ when the triangle is equilateral is 1.128..., which is close to the maximum.

Level 4

It seems that in this case we always get a degenerate straight line triangle.

What is the equation of the line AC joining $(\cos(2a), \sin(2a))$ to (1,0)? This is $y(\cos 2a - 1) = (x - 1)\sin 2a$.

Is D always on this line? We need to check if $(\cos^2 a, \cos a \sin a)$ is always on AC, which it is.

The locus of D is $r = \cos\theta$, which is a circle.

Level 5

We note here that as a tends towards $\frac{\pi}{2}$, the height of the triangle tends to ∞.

At the same time, however, the base of the triangle tends to 0. What happens to the area of the triangle in this situation; is it bounded? We have

$$A = 2 \times \left(\frac{1}{2}\cos a \tan a \sin a\right) - \frac{1}{2}\cos^2 a \sin(2a) = \sin^2 a - 2\sin a \cos^3 a.$$

At $a = \frac{\pi}{2}$, $A = 1$, and we can see the area of the triangle does remain bounded.

This risp is on page 36; Home Page

Further Risp 14: Three Simultaneous Equations: Notes

Topic: matrices, determinants, sequences, simultaneous equations, identities
Type of task: introductory
Preliminary knowledge required: solving simultaneous equations
Computing needs: calculator with 3×3 matrix capability or CAS

Level 1

If a square matrix is singular, then its inverse does not exist, and this happens if and only if its determinant is zero. If a square matrix has two columns equal, or two rows equal, then it is singular. We can also add or subtract any row of a matrix from any other row without changing the matrix's determinant (the same applies to columns). If one row is a multiple of another row, then the matrix is singular (the same applies to columns). Using these rules judiciously, we can often quickly show a matrix to be singular.

Level 2

Given the system of equations $\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ for some choice of a, b and c, we can use computing power to multiply both sides on the left by the inverse of $\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$ and we find the solution $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. We can see why this is; the right-hand side is $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ which is the same as the first column of the 3×3 matrix.

Level 3

Let's try a general arithmetic sequence $a, a+d, a+2d$, ... for some choice of a and d, which gives us

$$\begin{pmatrix} a & a+d & a+2d \\ a+4d & a+5d & a+6d \\ a+8d & a+9d & a+10d \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+3d \\ a+7d \\ a+11d \end{pmatrix}$$

to solve. Our computer, however, tells us this 3×3 matrix is singular, for all a and d in fact.

Why? If we take the top row from the second, we get a row of elements equal to $4d$.

If we then take the first row from the third, we get a row of elements equal to $8d$.

Now taking the revised second row from the third gives two rows the same, and so the determinant of the matrix is zero.

Let's try a general geometric sequence a, ar, ar^2, ..., which gives us

$$\begin{pmatrix} a & ar & ar^2 \\ ar^4 & ar^5 & ar^6 \\ ar^8 & ar^9 & ar^{10} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ar^3 \\ ar^7 \\ ar^{11} \end{pmatrix}$$

to solve. Once again we find this 3×3 matrix is singular for all a and r.

Why? Because the second column is r times the first.

Level 4

Let's try now the general Fibonacci sequence when $n = 2$;

$$a, b, a+b, a+2b, 2a+3b, ...$$

We have

$$\begin{pmatrix} a & b & a+b \\ 2a+3b & 3a+5b & 5a+8b \\ 13a+21b & 21a+34b & 34a+55b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+2b \\ 8a+13b \\ 55a+89b \end{pmatrix} \text{ to solve.}$$

Once again we find that the determinant is zero. If we take the second column from the third, then the first and third columns are the same.

Things change, however, if we look at the sequence starting with $1, 1, 1,$... where the next term is the sum of the previous three.

Here we have $\begin{pmatrix} 1 & 1 & 1 \\ 5 & 9 & 17 \\ 57 & 105 & 193 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 31 \\ 355 \end{pmatrix}$.

The 3×3 matrix has an inverse, $\begin{pmatrix} 3/2 & 11/4 & -1/4 \\ -1/8 & -17/4 & 3/8 \\ -3/8 & 3/2 & -1/8 \end{pmatrix}$, and when we premultiply $\begin{pmatrix} 3 \\ 31 \\ 355 \end{pmatrix}$ by this, we get $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Could we generalise this? Take the sequence that starts $a, b, c, a+b+c, ...$ where the next term is the sum of the previous three. We have

$$\begin{pmatrix} a & b & c \\ a+2b+2c & 2a+3b+4c & 4a+6b+7c \\ 13a+20b+24c & 24a+37b+44c & 44a+68b+81c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= \begin{pmatrix} a+b+c \\ 7a+11b+13c \\ 81a+125b+149c \end{pmatrix}.$$

Our computer (using CAS) gives something ferocious for the inverse of the 3×3 matrix on the left hand side, but when this pre-multiplies $\begin{pmatrix} a+b+c \\ 7a+11b+13c \\ 81a+125b+149c \end{pmatrix}$, we get $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, which is independent of a, b and c. We can see from the definition of the sequence why this solution works.

Level 5

How about the sequence of squares? We have

$$\begin{pmatrix} 1 & 4 & 9 \\ 25 & 36 & 49 \\ 81 & 100 & 121 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16 \\ 64 \\ 144 \end{pmatrix}.$$

We get a solution; the inverse of the matrix on the left is

$$\begin{pmatrix} 17/16 & -13/16 & 1/4 \\ -59/32 & 19/16 & -11/32 \\ 13/16 & -7/16 & 1/8 \end{pmatrix},$$

and when we multiply $\begin{pmatrix} 16 \\ 64 \\ 144 \end{pmatrix}$ by this, we get $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$.

How general is this result? We aim to solve

$$\begin{pmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ (a+4)^2 & (a+5)^2 & (a+6)^2 \\ (a+8)^2 & (a+9)^2 & (a+10)^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (a+3)^2 \\ (a+7)^2 \\ (a+11)^2 \end{pmatrix}.$$

Somewhat miraculously, our computer can confirm (using CAS) that the solution for this is always $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$ (this does not work for cubes or higher powers). What are effectively seeing here is the identity

$$(a+n)^2 + 3(a+(n+2))^2 = 3(a+(n+1))^2 + (a+(n+3))^2.$$

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Further Risp 15: Matrix Logic: Notes

Topic: matrices
Type of task: revision
Preliminary knowledge required:
Computing needs:

Level 1

1. $A : \det(P) = \det(Q) = k, k \in \mathbb{Z}, B : \det(PQ)$ is a perfect square.

We know $\det(PQ) = \det(P)\det(Q)$, so $\det(P) = \det(Q) = k$ for integer $k \Rightarrow \det(PQ)$ is a perfect square and $A \Rightarrow B$.

But $\det(P) = \sqrt{2}, \det Q = \sqrt{8}$ gives $\det(PQ) = 4$, so $B \nRightarrow A$.

This argument works just as well for 3×3 matrices as for 2×2 matrices.

2. $A : P$ represents a rotation about O, $B : P$ represents a reflection in a line (for 2×2) or plane (for 3×3) through the origin.

If A is true, P will have determinant 1, whereas if B is true, P will have determinant -1.

Thus A rules out B, and B rules out A.

3. $A : PQ$ is singular, $B : P$ or Q or both are singular.

We have $\det(PQ) = \det P \det Q$, so $\det(PQ) = 0 \Rightarrow \det P = 0$ or $\det Q = 0$ (or both).

Similarly $\det P = 0$ or $\det Q = 0 \Rightarrow \det(PQ) = 0$.

Thus $A \Leftrightarrow B$.

Level 2

4. $A: PQ = QP, B: P$ and Q both represent rotations about the origin.

In two dimensions, certainly $B \Rightarrow A$, but the converse is not true (for example, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$ but neither $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ represent rotations) so $A \not\Rightarrow B$.

In three dimensions, P and Q could represent rotations about different axes, so $A \not\Rightarrow B$, and $B \not\Rightarrow A$.

5. P is not singular. $A: \det(kP) = k\det(P), k > 0, B: k = 0$ or 1.

In two dimensions, clearly $B \Rightarrow A$. To find kP every element of P is multiplied by k, which multiplies the determinant by k^2. So

$$\det(kP) = k\det(A) \Leftrightarrow k = k^2 \Leftrightarrow k = 0 \quad \text{or} \quad 1.$$

So here we have $A \Leftrightarrow B$.

In three dimensions, again to find kP every element of P is multiplied by k, which multiplies the determinant this time by k^3. So

$$\det(kP) = k\det(A) \Leftrightarrow k = k^3 \Leftrightarrow k = 0 \quad \text{or} \quad \pm 1.$$

So here $B \Rightarrow A$ but $A \not\Rightarrow B$.

6. $A: P$ represents a rotation, $B: \det(P) = 1$.

With a rotation, the area scale factor is 1, so the determinant of P must be 1. But there are matrices with determinant 1 that don't represent rotations; $\begin{pmatrix} 5 & 3 \\ 8 & 5 \end{pmatrix}$, for example. In both two and three dimensions, $A \Rightarrow B$ but $B \not\Rightarrow A$.

Level 3

7. $A : P$ represents a rotation through θ about the origin, $0 < \theta < \pi$, $B : P$ has no real eigenvectors.

In two dimensions, if A is true there will be no real eigenvectors, since every starting vector changes in direction. So $A \Rightarrow B$. If P has no real eigenvectors, then its characteristic equation $t^2-(a+d)t+ad-bc = 0$ must have complex roots, which is true if and only if $(a+d)^2 < 4(ad-bc) \Leftrightarrow (a-d)^2 < -4bc$. The matrix $\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$ satisfies this inequality, but this matrix does not represent a rotation. Thus $B \not\Rightarrow A$.

In three dimensions, if A is true then there will be an axis of rotation which will supply a direction for one set of eigenvectors. It is worth noting that since every cubic equation with real coefficients has at least one real root, any cubic characteristic equation with real coefficients will have at least one real root (eigenvalue), and so it is impossible for there to be (for a 3×3 matrix) no real eigenvectors. Thus B is false, regardless of whether A is true or not.

8. $A : P$ is the zero matrix, $B : P$ has no real eigenvalues.

For both the two-dimensional and three-dimensional cases, the zero matrix has the repeated real eigenvalue 0, so A rules out B, and B rules out A.

9. P and Q are not singular. $A : \det(P+Q) = \det P + \det Q, B : P = Q$.

Let's take the 2×2 case first. We have

$$B \Rightarrow \det(P+Q) = \det(2P) = 4\det(P), \det P + \det Q = 2\det P,$$

so B rules out A (since $\det P \neq 0$).

It is not too hard to find matrices P and Q so that A is true but B is not, for example $P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. But if A is true, B cannot be true, so A rules out B.

So here we have A rules out B and B rules out A.

The 3×3 case goes in much the same way. If $P = \begin{pmatrix} 3 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$ and $Q = \begin{pmatrix} -3 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$ then $\det(P+Q) = \det P + \det Q = 0$.

Level 4

10. P and Q are not singular. $A : (PQ)^{-1} = P^{-1}Q^{-1}, B : PQ = QP$.

We know that $(PQ)^{-1} = Q^{-1}P^{-1}$ whenever P and Q are not singular. So

$$\begin{aligned} PQ = QP &\Leftrightarrow Q^{-1}P^{-1}PQ = Q^{-1}P^{-1}QP \\ &\Leftrightarrow I = Q^{-1}P^{-1}QP \\ &\Leftrightarrow P^{-1}Q^{-1} = Q^{-1}P^{-1}QPP^{-1}Q^{-1} \\ &\Leftrightarrow P^{-1}Q^{-1} = Q^{-1}P^{-1}. \end{aligned}$$

So here $A \Leftrightarrow B$.

11. P and Q are 2×2 reflection matrices. $A : PQ = QP, B : P = Q$.

It is clear that $B \Rightarrow A$, since in this case $PQ = QP = I$.

It is possible to find reflection matrices P and Q so that $PQ = QP$ but $P \neq Q$. Consider $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, a reflection in $y = x$, and $Q = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, a reflection in $y = -x$, so that $PQ = QP = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus $A \not\Rightarrow B$.

12. T is an enlargement with scale factor $k \neq 1$. $A : T$ can be represented by a matrix $P, B : T$ has its centre at the origin.

Certainly B implies A; the matrix $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ or $\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$ is the required one.

If T does not have its centre at the origin, then there are two points that are invariant under the enlargement (its centre and the origin). But the distance between these two points is multiplied by k under T, and so T cannot be an enlargement. Thus we have (for both two and three dimensions) $A \Leftrightarrow B$.

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Further Risp 16: Periodic Recurrence Relations: Notes

Topic: recurrence relations
Type of task: introductory
Preliminary knowledge required: none
Computing needs: Excel spreadsheet

Level 1

The order-3 recurrence $z = -y - x - w$ has this for its first few terms; $w, x, y, -y - x - w, w, x, y, -y - x - w, w ...$

This recurrence is order-3, period-3. We can easily extend this to the order-n case.

Level 2

The order-1 recurrence $u_{n+1} = \dfrac{u_n + 1}{u_n - 1}$ with $u_1 = 2$ gives the sequence 2, 3, 2, 3, 2..., which is periodic, with period 2.

Putting $u_1 = x$ gives the sequence $x, \dfrac{x+1}{x-1}, x, \dfrac{x+1}{x-1}, x, \dfrac{x+1}{x-1}$, ..., which is periodic for all $x \neq 1$, with period 2. This type of recurrence, that is periodic for almost all values of x, is called *globally periodic*, and is relatively rare.

Level 3

The Excel spreadsheet enables us to check a large number of possibilities. For order 1, it is possible to find recurrences of period 1, 2, 3, 4 and 6, while for order 2 we can find recurrences of period 2, 3, 4, 5 and 6.

Period	Order 1 example $y =$	Order 2 example $z =$
1	x	$\times$
2	$-x$	x
3	$1-\frac{1}{x}$	$-x-y$
4	$1-\frac{1}{2x}$	$\frac{1}{x}$
5	$\times$	$\frac{y+1}{x}$
6	$1-\frac{1}{3x}$	$\frac{y}{x}$

Level 4

It's possible to prove that order-1, period-5 is impossible if we insist on using integers for the coefficients, but this proof is hard! If we allow rational numbers for the coefficients, order-1, period-5 is still impossible. There are some values of n beyond $n = 6$ where order-2 periodic recurrences can be found. If we allow the coefficients to be irrational, then any period is possible; for order-2, for example, the recurrence $z = ky - x$ can have any period as k varies through the real numbers.

This kind of periodic recurrence relation is currently a hot topic amongst researchers. Perhaps the most famous example is

$$x, y, \frac{y+1}{x}, ...$$

as popularised by R.C.Lyness in the 1940s (the table gives this as period-5). To work out the first few terms by hand is great algebra practice and remains a hugely pleasing thing to do!

This risp is on page 42; Home Page

Further Risp 17: Summing Powers: Notes

Topic: summing powers formulae
Type of task: consolidation
Preliminary knowledge required: basic formulae and sigma sign
Computing needs: none

Level 1

We're given that $\displaystyle\sum_1^n ar+b = \frac{an(n+1)}{2}+bn = \frac{an^2}{2}+\frac{a+2b}{2}n = An^2+Bn.$

Given a and b, we can always find A and B, and vice versa.

If A and B are integers, then $a = 2A$ and $b = B - A$, and a and b are both integers.

The converse is not true, since if $a = 3, b = 1$, then A and B are not integers. But if a and b are both integers with a even, then A and B will both be integers.

Level 2

We have $\displaystyle\sum_1^n (cr+d)r = c\sum_1^n r^2 + d\sum_1^n r = n^3\frac{c}{3} + n^2\frac{c+d}{2} + n\frac{c+3d}{6}$

$= Cn^3 + Dn^2 + En$ for all positive integers n.

Thus $c = 3C, c+d = 2D, 6E = c+3d = 3C+6D-9C \implies D = C+E.$

So this time there is a condition on the values of C, D and E.

But if $D = C+E$, then $c = 3C$, and $d = 2D - 3C$, and so if C, D and E are integers, c and d will be too.

C, D and E are integers if $6|c$ and $2|d$.

Level 3

We have

$$\begin{aligned}\sum_1^n (fr+g)(r+1) &= f\sum_1^n r^2 + (f+g)\sum_1^n r + gn \\ &= n^3\frac{f}{3} + n^2\frac{2f+g}{2} + n\frac{4f+9g}{6} \\ &= Fn^3 + Gn^2 + Hn.\end{aligned}$$

So $f = 3F, 2f + g = 2G$, and $4f + 9g = 6H$.

So if we are given integers F, G and H, can we always find integer f and g that satisfy these equations?

We have $F = \dfrac{f}{3}, G = f + \dfrac{g}{2}, H = \dfrac{2f}{3} + \dfrac{3g}{2}$, so if $3|f$ and $2|g$ then F, G and H are integer.

Going the other way, $f = 3F, g = 2G - 6F$, which leaves $H = 3G - 7F$, so we have a condition on H for our starting equation to be true at all.

If we have F, G integer and $H = 3G - 7F$, then f and g will exist and be integer.

Level 4

We know $\displaystyle\sum_1^n (jr+k)(r+1)r = j\sum_1^n r^3 + (j+k)\sum_1^n r^2 + k\sum_1^n r$

$\displaystyle = j\frac{n^2(n+1)^2}{4} + (j+k)\frac{n(n+1)(2n+1)}{6} + k\frac{n(n+1)}{2}$

$\displaystyle = n^4\frac{j}{4} + n^3\frac{5j+2k}{6} + n^2\frac{3j+2k}{4} + n\frac{j+4k}{6}.$

$= Jn^4 + Kn^3 + Ln^2 + Mn$ for all positive integers n.

Thus on multiplying out, $J = \dfrac{j}{4}, K = \dfrac{5j}{6} + \dfrac{k}{3}, L = \dfrac{3j}{4} + k, M = \dfrac{j}{8} + \dfrac{2k}{3}.$

We can see that for J, K, L and M to be integer, we need $24|j$ and $3|k$.

We also have that $j = 4J, k = 3K - 10J \Rightarrow L = 3K - 7J, M = 2K - 6J$ (together these imply that $L + J = K + M$).

So there is a condition on L and another on M for our starting equation to hold.

But if J and K are integers and these conditions hold, j and k will be integers too.

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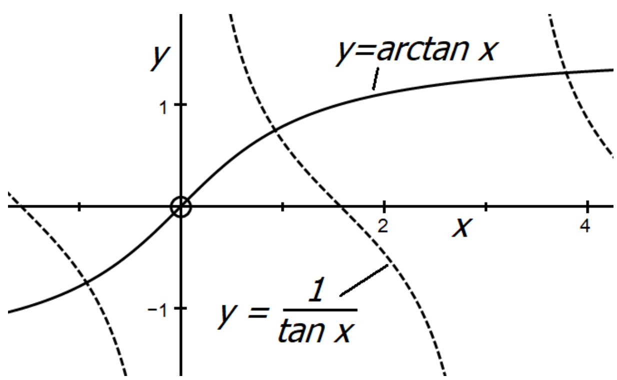
Further Risp 18: Do you really mean 'arctan'? Notes

Topic: numerical methods
Type of task: consolidation
Preliminary knowledge required: Newton-Raphson, the rearrangement method, and linear interpolation
Computing needs: a graphing package

Level 1

Drawing the graphs of $y = \arctan x$ and $y = \dfrac{1}{\tan x}$ shows that there are roots, and the positive root closest to the origin is at approximately $x = 1$.



We can rearrange $\arctan x = \dfrac{1}{\tan x}$ to $x = \tan(\cot x)$, but the curve $y = \tan(\cot x)$ has a gradient larger than 1 in size at the root, and so the iteration fails.

There is a neat method that we can employ here; add mx to both sides of $x = \tan(\cot x)$.

This gives us $(m+1)x = \tan(\cot x) + mx$ and so $x = \dfrac{\tan(\cot x) + mx}{m+1}$.

We can draw $y = \dfrac{\tan(\cot x) + mx}{m+1}$ and vary m so that the size of this function's gradient at the root is less than 1. This means our iteration will now converge.

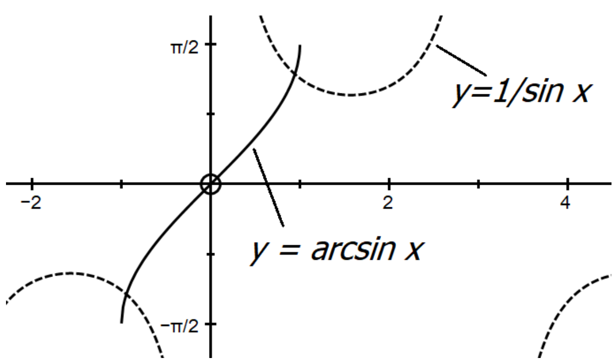
It works well to choose $m = 2$ and $x_0 = 1$ here, which yields the root 0.9284(4s.f.). Excel (or the ANS button on your calculator) is useful in giving this root swiftly.

B1 f_x =(TAN(COT(A1))+2*A1)/3

	A	B	C	D	E
1	1	0.91593385		0	
2	0.915934	0.932493451			
3	0.932493	0.927194524			
4	0.927195	0.928759818			
5	0.92876	0.928285114			
6	0.928285	0.928427973			
7	0.928428	0.92838488			
8	0.928385	0.928397869			
9	0.928398	0.928393953			

Level 2

Drawing the graphs of $y = \arcsin x$ and $y = \dfrac{1}{\sin x}$ shows that there are roots for $\arcsin x = \dfrac{1}{\sin x}$, and the positive root closest to the origin is at approximately $x = 0.9$.



The Newton-Raphson iteration tells that given an approximation x_n to a root of $f(x) = 0$, we can (usually) find a better one given by

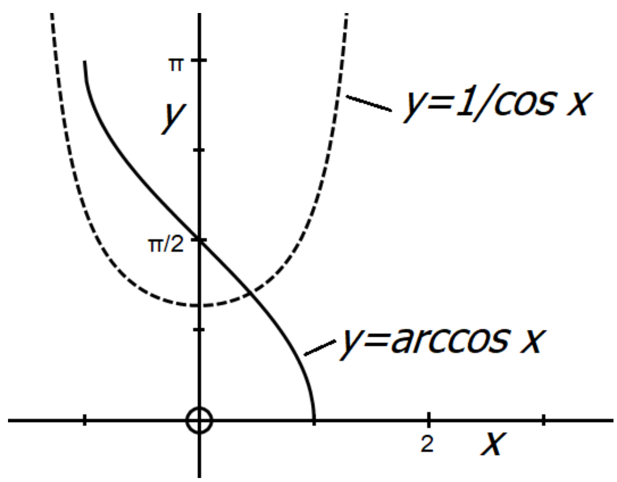
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If the first guess at the root is reasonably close, this generally converges fast. So here we have

$$x_0 = 0.9, x_{n+1} = x_n - \frac{\arcsin x_n - \operatorname{cosec} x_n}{\frac{1}{\sqrt{1-x_n^2}} + \cos x_n (\operatorname{cosec} x_n)^2},$$

and Excel (or the ANS button on our calculator) comes to our aid again, giving us the answer 0.9440(4s.f.).

Level 3



This graph tells us that $\arccos x = \dfrac{1}{\cos x}$ has exactly one solution.

Linear interpolation on an interval $[a, b]$ (where $f(a)$ and $f(b)$ are of opposite sign) that contains a root of $f(x) = 0$ replaces the curve with a straight line, and generates this estimate of the root;

$$\alpha = \frac{a|f(b)| + b|f(a)|}{|f(b)| + |f(a)|}.$$

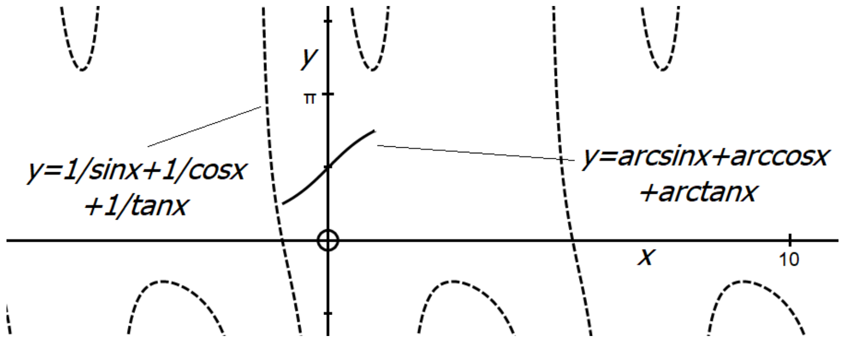
Calculate $f(\alpha)$; is this positive or negative? The value α then becomes one of the endpoints of the new (reduced) interval that still sees a change of sign.

We can take $[0, 1]$ as our starting interval, where $f(x) = \arccos x - \dfrac{1}{\cos x}$. Excel gives us this;

a	*b*	*alpha*	*abs(f(a))*	*abs(f(b))*	*f(alpha)*
0	1	0.235709	0.570796	1.850816	0.304410656
0.235709237	1	0.34366	0.304411	1.850816	0.157881335
0.343659963	1	0.395248	0.157881	1.850816	0.080919479
0.395247554	1	0.42058	0.080919	1.850816	0.041243767
0.420580349	1	0.433211	0.041244	1.850816	0.020964691
0.433210739	1	0.439559	0.020965	1.850816	0.010642323
0.439559006	1	0.442763	0.010642	1.850816	0.005398746
0.442763158	1	0.444384	0.005399	1.850816	0.002737805
0.444383865	1	0.445205	0.002738	1.850816	0.001388155
0.445204542	1	0.44562	0.001388	1.850816	0.000703778
0.44562034	1	0.445831	0.000704	1.850816	0.000356791
0.445831064	1	0.445938	0.000357	1.850816	0.000180877
0.445937873	1	0.445992	0.000181	1.850816	9.16955E-05
0.445992016	1	0.446019	9.17E-05	1.850816	4.64847E-05
0.446019462	1	0.446033	4.65E-05	1.850816	2.35652E-05
0.446033375	1	0.44604	2.36E-05	1.850816	1.19462E-05
0.446040428	1	0.446044	1.19E-05	1.850816	6.05605E-06
0.446044004	1	0.446046	6.06E-06	1.850816	3.07008E-06
0.446045816	1	0.446047	3.07E-06	1.850816	1.55635E-06
0.446046735	1	0.446047	1.56E-06	1.850816	7.88983E-07
0.446047201	1	0.446047	7.89E-07	1.850816	3.9997E-07

Here the curvature of the curve is such that $f(\alpha)$ is always positive using this procedure. We arrive at the value of 0.4460(4s.f.) for the root.

Level 4



The graphs of $y = \arcsin x + \arccos x + \arctan x$ (solid) and $y = \dfrac{1}{\sin x} + \dfrac{1}{\cos x} + \dfrac{1}{\tan x}$ (dashed) are drawn in the diagram above. It seems that the two curves almost meet, but not quite! Thus the equation

$$\arcsin x + \arccos x + \arctan x = \frac{1}{\sin x} + \frac{1}{\cos x} + \frac{1}{\tan x}$$

has no roots.

Some might ask, 'Are these techniques really needed today?' Given the power of our modern graphing programs, we can simply zoom in on any root to find it to as much accuracy as we need. It may be that now these approximate methods above, linear interpolation, Newton Raphson and rearrangement, are historically interesting but less so practically. That said, the convergence or non-convergence of sequences like these remains an excellent thing to know about.

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Further Risp 19: Roots of Cubic Equations: Notes

Topic: sum and product of roots
Type of task: consolidation
Preliminary knowledge required: formulae for sum and product of roots
Computing needs: none

Level 1

For a cubic equation $ax^3 + bx^2 + cx + d = 0$ with roots α, β and γ,

$$\alpha + \beta + \gamma = -\frac{b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a},$$

$$\alpha\beta\gamma = -\frac{d}{a}.$$

Level 2

1. $x + 1 = 0$? Solving this is no problem.

2. $x^2 + x + 1 = 0$? We can use the quadratic formula here to find the two (complex) roots.

3. $x^3 + x^2 + x + 1 = 0$? Using the factor theorem, we can see that $x + 1$ is a factor of the left hand side, giving us $(x + 1)(x^2 + 1) = 0$, and we have no problem in finding the roots.

4. $x^3 + x^2 + x + 2 = 0$? Tricky. There is a cubic formula, similar to the quadratic one, but much more complicated. We could use this to find exact roots.

5. $x^3 + x^2 + x = 0$? Here we can take out x as a factor, giving $x(x^2 + x + 1) = 0$. We now have $x = 0$ as a root and we can use the quadratic formula on the other bracket.

Level 3

6. $x^4 + x^3 + x^2 + x + 1 = 0$? The factor theorem is no help. There is a quartic formula akin to the quadratic one which can use, but it's not easy.

7. $x^4+x^2+1 = 0$? This is a quadratic in x^2, and we can use the quadratic formula here.

8. $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$? Using the factor theorem, we can see that $x+1$ is a factor of the left hand side, giving us $(x+1)(x^4+x^2+1) = 0$. We have already spotted that the second bracket is a quadratic in disguise, and we can use the quadratic formula here to find the roots.

9. $x^5 + x^4 + x^3 + x^2 + x + 2 = 0$? There is no exact formula possible for quintic or higher order equations, as shown by Abel and Ruffini. We would have to use approximate methods here.

Level 4

We need four non-zero integers $p, q, r, s \geq 2$, with no common factor shared between all four, so that $ps - qr = 0$. Let's choose

$$p = 10, q = 5, r = 4, s = 2.$$

This gives us the equation $10x^3+5x^2+4x+2 = 0$ to solve. Let's suppose the roots here are α, β and γ. We are asked to find the equation with roots $\alpha+\beta, \gamma+\alpha$ and $\beta+\gamma$.

We know that $\alpha + \beta + \gamma = -\dfrac{5}{10}, \alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{4}{10}, \alpha\beta\gamma = -\dfrac{2}{10}$.

This means that $(\alpha+\beta) + (\gamma+\alpha) + (\beta+\gamma) = 2(\alpha+\beta+\gamma) = -1$. Also

$$\begin{aligned}
&(\alpha+\beta)(\gamma+\alpha) + (\beta+\gamma)(\alpha+\beta) + (\gamma+\alpha)(\beta+\gamma) \\
&= \alpha^2 + \beta^2 + \gamma^2 + 3(\alpha\beta + \beta\gamma + \gamma\alpha) \\
&= (\alpha+\beta+\gamma)^2 + (\alpha\beta + \beta\gamma + \gamma\alpha) \\
&= \frac{13}{20}.
\end{aligned}$$

Additionally

$$\begin{aligned}&(\alpha+\beta)(\gamma+\alpha)(\beta+\gamma)\\=&2\alpha\beta\gamma+(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha+\beta+\gamma)-3\alpha\beta\gamma\\=&(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha+\beta+\gamma)-\alpha\beta\gamma,\end{aligned}$$

and when we substitute in, we find this is zero. Thus the equation with roots $\alpha+\beta, \gamma+\alpha$ and $\beta+\gamma$ is

$$x^3+x^2+\frac{13}{20}x=0.$$

Clearly one of the roots is 0, and we can find the others by using the quadratic formula on $x^2+x+\frac{13}{20}=0$, or $20x^2+20x+13=0$.

So our roots $\alpha+\beta, \gamma+\alpha$ and $\beta+\gamma$ are $0, -\frac{1}{2}+\sqrt{\frac{2}{5}}i, -\frac{1}{2}-\sqrt{\frac{2}{5}}i$.

Say $\alpha+\beta=0$, then on adding the other two equations, $2\gamma=-1$, so $\gamma=-\frac{1}{2}$. Thus $\alpha=\sqrt{\frac{2}{5}}i$, and $\beta=-\sqrt{\frac{2}{5}}i$, and we have solved our starting cubic equation.

Level 5

Can you find an alternative way to solve $px^3+qx^2+rx+s=0$ when $ps-qr=0$?

If $ps-qr=0$, we can write our equation as $px^3+qx^2+rx+\frac{qr}{p}=0$, assuming $p\neq 0$ (this case is easily dealt with).

The left hand side factorises into $\frac{(px+q)(px^2+r)}{p}$, and we can find the roots from here.

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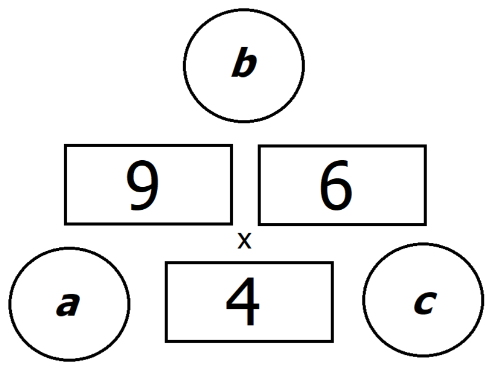
Further Risp 20: Matrix Arithmagon: Notes

Topic: matrices
Type of task: introductory
Preliminary knowledge required:
matrix multiplication and inverses for 2×2
Computing needs: a calculator for matrix multiplication

Level 1



We can see that $ab = 9, bc = 6$ and $ca = 4$. Dividing the first two equations gives us $\frac{a}{c} = \frac{3}{2}$, and multiplying this by the third equation gives us $a^2 = 6$.

Thus $(a, b, c) = \left(\sqrt{6}, \frac{3\sqrt{6}}{2}, \frac{2\sqrt{6}}{3}\right)$, or $\left(-\sqrt{6}, -\frac{3\sqrt{6}}{2}, -\frac{2\sqrt{6}}{3}\right)$.

So in general there are two solutions, one the negation of the other (there may be no real solutions at all; for example, if all the rectangles are negative).

Level 2

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$. Multiplying out, we find that

$$a^2 + bc = 2, b(a + d) = 1, c(a + d) = 2, bc + d^2 = 3.$$

We have four equations in four unknowns, so we hope to find a, b, c and d here. We'll first try to find a, c and d in terms of b.

Dividing the middle two equations (assuming $b \neq 0$) gives

$$\frac{c}{b} = 2 \Rightarrow c = 2b.$$

Substituting into the third equation gives $a + d = \dfrac{1}{b} \Rightarrow d = -a + \dfrac{1}{b}$.

Subtracting the first equation from the last gives

$$d^2 - a^2 = (d+a)(d-a) = \frac{1}{b}(d-a) = 1.$$

Thus $d = a + b$, and $2a = -b + \dfrac{1}{b} \Rightarrow a = -\dfrac{b}{2} + \dfrac{1}{2b}$.

Now from the first equation, $\left(-\dfrac{b}{2} + \dfrac{1}{2b}\right)^2 + b(2b) = 2$.

Multiplying out and rearranging gives us

$$9b^4 - 10b^2 + 1 = 0 \Rightarrow (9b^2 - 1)(b^2 - 1) = 0.$$

Thus $b = \pm\dfrac{1}{3}, \pm 1$. From here we can find a, c and d.

This means there are four 'square roots' for the matrix $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$;

$$\frac{1}{3}\begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} -4 & -1 \\ -2 & -5 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -2 & -1 \end{pmatrix}.$$

Level 3

We know that $D = AB$ and $E = BC$, so $A = DB^{-1}$ and $C = B^{-1}E$.

Multiplying these gives $AC = D(B^{-1})^2E = F$.

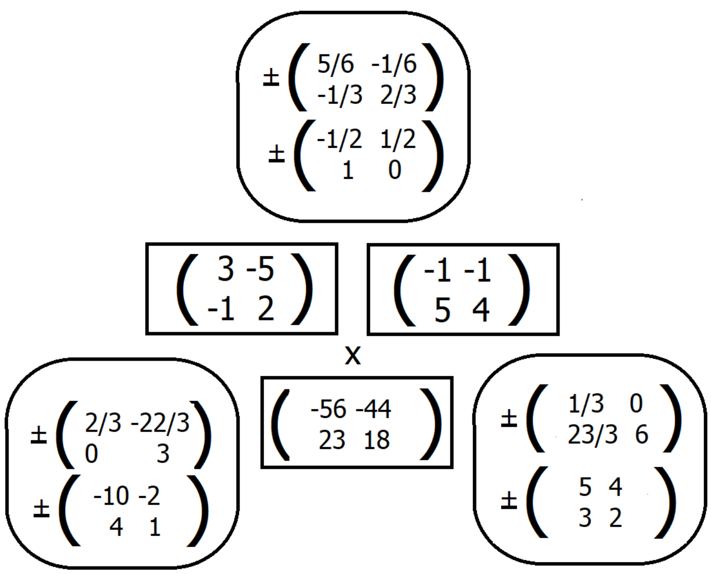
Thus $(B^{-1})^2 = D^{-1}FE^{-1}$, which means that given non-singular D, E and F, we can find the possibilities for B^{-1} and from there for B, and then for A and C.

Given our earlier work, we are expecting four solutions for A, B, C and D.

Level 4

Using the above technique here, we find $(B^{-1})^2 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$.

So we know the four possibilities for B^{-1}, and thus for B, and we arrive at the four possible solutions below;



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Further Risp 21: Ordering Loops: Notes

Topic: polar coordinates
Type of task: consolidation
Preliminary knowledge required:
area in polar coordinates, standard polar curves
Computing needs: a graphing package

What curves can we make? We have these eleven (there may be more), that we can tell make a loop symmetrical about the principal line, since anything of the form $r = p\cos(q\theta)$ or $r = p\cos(q\theta^2)$ or $r = p(\cos(q\theta))^2$ will provide this for us.

n	Equation	$\theta = 0 \Rightarrow r =$	Tangent at O	Type of curve
1	$r = \cos(2^2\theta)$	1	$\theta = \pi/8 = 0.3927$	rose
2	$r = \cos(2\theta^2)$	1	$\theta = \sqrt{\pi/4} = 0.8862$	no name
3	$r = (\cos 2\theta)^2$	1	$\theta = \pi/4 = 0.7854$	rose
4	$2r = (\cos\theta)^2$	0.5	$\theta = \pi/2 = 1.571$	rose
5	$2r = \cos(\theta^2)$	0.5	$\theta = \sqrt{\pi/2} = 1.253$	no name
6	$2^2 r = \cos\theta$	0.25	$\theta = \pi/2 = 1.571$	circle
7	$r = 2(\cos\theta)^2$	2	$\theta = \pi/2 = 1.571$	rose
8	$r = 2\cos(\theta^2)$	2	$\theta = \sqrt{\pi/2} = 1.253$	no name
9	$r = \cos((2\theta)^2)$	1	$\theta = \sqrt{\pi/8} = 0.6267$	no name
10	$r = 2^2\cos\theta$	4	$\theta = \pi/2 = 1.571$	circle
11	$r = (2\cos\theta)^2$	4	$\theta = \pi/2 = 1.571$	rose

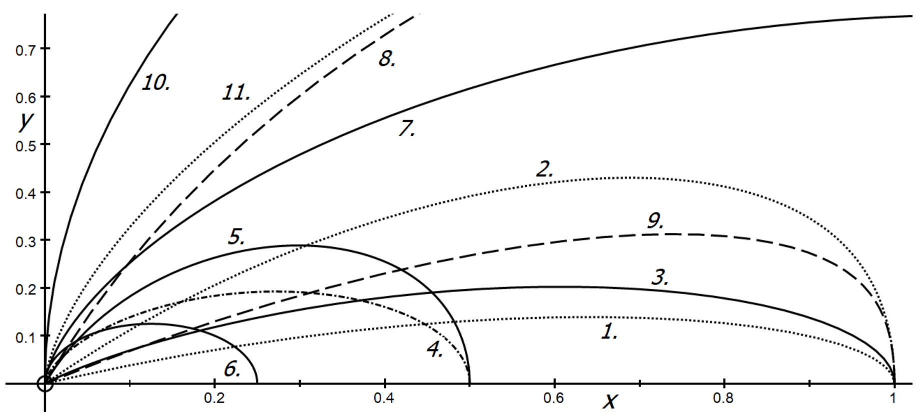
Can we now create a list by ordering these according to loop area? It seems highly likely that ordering by column 3 is judicious, and then ordering by angle of tangent at the origin is likely to provide a second filter for our results. This rough method suggests we have, from largest to smallest,

$$11, 10, 7, 8, 2, 3, 9, 1, 4, 5, 6.$$

We can also come up with lots of equations that fail to make a symmetrical loop.

n	Equation	Type of curve
12	$r = (\cos 2)^2\theta$	spiral
13	$(\cos 2)^2 r = \theta$	spiral
14	$\cos(2^2)r = \theta$	spiral
15	$\cos\theta r = 2^2$	straight line
16	$(\cos\theta)^2 r = 2$	no name
17	$\theta^2 r = \cos 2$	no name
18	$\theta r = \cos(2^2)$	no name
19	$\theta r = (\cos 2)^2$	no name
20	$\cos r = (2\theta)^2$	no name

Let's turn to the computer now, and draw half of the loop for each of our first 11 equations.



The order seems clear, except for perhaps 5. and 1. Can we compare 5. and 1. by direct calculation?

For 1. the area for the whole of the loop is $2\int_0^{\pi/2} \frac{1}{2}(\cos(4\theta))^2 d\theta$.

Using the identity $(\cos\theta)^2 = \dfrac{\cos 2\theta + 1}{2}$, this solves to give an area of 0.1963(4s.f.).

For 5. the area for the whole of the loop is $2\int_0^{\pi/2}\frac{1}{2}(\frac{1}{2}\cos(\theta^2))^2 d\theta$. Integrating $\cos^2(\theta^2)$ algebraically is impossible; we will have to use approximate methods. These give 0.2152(4s.f.).

n	Equation	Area of whole loop (4s.f.)
1	$r=\cos(2^2\theta)$	0.1963
2	$r=\cos(2\theta^2)$	0.6088
3	$r=(\cos 2\theta)^2$	0.2945
4	$2r=(\cos\theta)^2$	0.1473
5	$2r=\cos(\theta^2)$	0.2153
6	$2^2r=\cos\theta$	0.04909
7	$r=2(\cos\theta)^2$	2.356
8	$r=2\cos(\theta^2)$	3.444
9	$r=\cos((2\theta)^2)$	0.4305
10	$r=2^2\cos\theta$	12.57
11	$r=(2\cos\theta)^2$	9.425

So now we have the true order for the areas, from large to small:

$$11, 10, 8, 7, 2, 9, 3, 5, 1, 4, 6.$$

Theory tells us that area contained by $r=f(\theta)$ and the lines $\theta=a$ and $\theta=b$ is $\frac{1}{2}\int_a^b(f(\theta))^2 d\theta$. What does that mean for the functions we have created? We need to find $\int p\cos^2(q\theta)d\theta$, or $\int p\cos^4(q\theta)d\theta$, or $\int p\cos^2(q\theta^2)d\theta$.

For $\int p\cos^2(q\theta)d\theta$, we need to use the identity $\cos^2(q\theta)=\dfrac{\cos(2q\theta)+1}{2}$.

For $\int p\cos^4(q\theta)d\theta$, we need to use this twice; $\cos^4(q\theta)=\left(\dfrac{\cos(2q\theta)+1}{2}\right)^2$, and then $\cos^2(2q\theta)=\dfrac{\cos(4q\theta)+1}{2}$.

The integral $\int p\cos^2(q\theta^2)d\theta$ can't be solved exactly, and we need to use numerical integration.

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Further Risp 22: The Vector Product: Notes

Topic: vector cross product
Type of task: introductory
Preliminary knowledge required: basic vector knowledge including scalar product, determinants
Computing needs: none

Level 1

We are told $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ is perpendicular to both $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

We know that two vectors $\underline{a}$ and $\underline{b}$ are perpendicular if and only if $\underline{a}.\underline{b} = 0$, where . is the scalar product.

So we can say $a_1c_1 + a_2c_2 + a_3c_3 = 0, b_1c_1 + b_2c_2 + b_3c_3 = 0$.

Multiply the first equation by b_1 and the second by a_1, and subtract. Now rearranging gives

$$c_3 = \frac{a_2b_1 - a_1b_2}{a_1b_3 - a_3b_1}c_2,$$

as long as the denominator is not zero. Substituting into one of the equations we can now find c_1 in terms of c_2, and so $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ becomes

$$\begin{pmatrix} \frac{a_3b_2 - a_2b_3}{a_1b_3 - a_3b_1}c_2 \\ c_2 \\ \frac{a_2b_1 - a_1b_2}{a_1b_3 - a_3b_1}c_2 \end{pmatrix}.$$

Now divide each element by c_2 and multiply each element by $a_1b_3 - a_3b_1$ (this won't change the direction of the vector). We finally arrive at

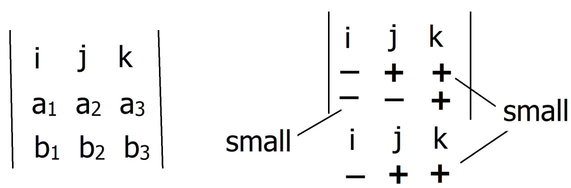
$$\begin{pmatrix} a_3b_2 - a_2b_3 \\ a_1b_3 - a_3b_1 \\ a_2b_1 - a_1b_2 \end{pmatrix}.$$

This is in fact $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$, so any vector perpendicular to our given vectors $\underline{a}$ and $\underline{b}$ must be a multiple of this vector.

Level 2

Can $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, where three of a_1, a_2, a_3, b_1, b_2 and b_3 are positive and three are negative, be $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ where c_1, c_2 and c_3 are all positive?

We know $\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$; imagine calculating this via Sarri's method.



To give ourselves the best chance for this determinant to be positive, we want the diagonals from top left to bottom right to be positive, and the ones from top right to bottom left to be negative. Choosing the positive and negative values as shown seems sensible, leaving just two values that we need to be small in size in relation to the others.

So yes, it is possible for c_1, c_2 and c_3 to all be positive here.

Level 3

Consider the region R created by the positive x, y and z-axes.

If $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ both lie in R, then the plane defined by these two vectors and the origin also passes through R.

The vector $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is perpendicular to the plane, and thus cannot lie in R when viewed as a position vector.

So the equation $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ cannot hold if all nine elements are to be positive.

There is a nice algebraic proof of this. We have

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

If the three elements on the right are all positive, then

$$a_2b_3 > a_3b_2 \Rightarrow \frac{a_2b_3}{b_2} > a_3.$$

We also have

$$a_3b_1 > a_1b_3 \Rightarrow a_3 > \frac{a_1b_3}{b_1} \Rightarrow \frac{a_2b_3}{b_2} > \frac{a_1b_3}{b_1} \Rightarrow a_2b_1 > a_1b_2.$$

But the final element c_3 tells us $a_2b_1 < a_1b_2$, which is a contradiction.

Level 4

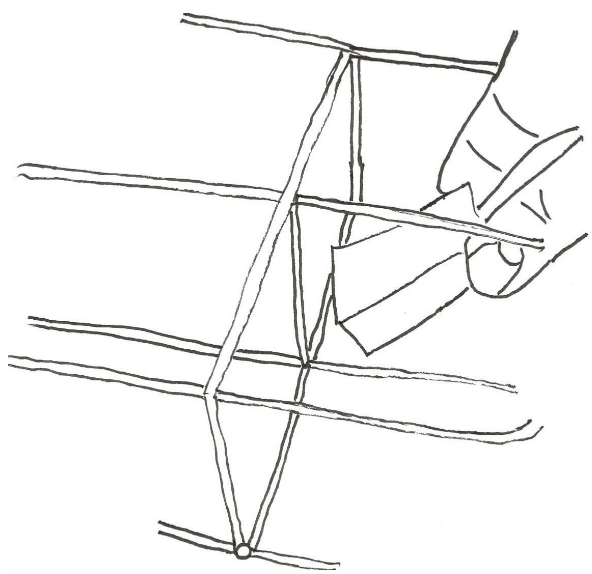
The planes $x = 0, y = 0$ and $z = 0$ divide three-dimensional space into eight regions. We can label the regions according to whether they are defined by the positive or negative directions for x, y and z, giving us

$$(+++), (++-), (+-+), (-++), (--+), (-+-), (+--), (---).$$

So $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ cannot lie in $(+,+,+)$ if $\underline{a}$ and $\underline{b}$ each lie in $(+,+,+)$. How many of c_1, c_2 and c_3 *can* be positive here?

We can first notice that $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$, so if a region *is* possible for some $\underline{a}$ and $\underline{b}$ with positive elements, the region diametrically opposite is also possible, which can be shown by simply swapping the order of the vectors $\underline{a}$ and $\underline{b}$ in the product.

If you happen to have a matchbox and a plastic box divided into four compartments, this helps with the visualisation here. The bottom of the matchbox is the plane for $\underline{a}$ and $\underline{b}$, and the short edge of the box gives the direction of $\underline{a} \times \underline{b}$.



This gives us all regions are possible for $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ except for $(+++)$ and $(---)$.

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Further Risp 23: The Matrix Puzzle: Notes

Topic: matrices
Type of task: consolidation/revision
Preliminary knowledge required: matrix terminology including eigenvalues/eigenvectors
Computing needs: calculator for multiplying matrices

Level 2

To get started, it's key to realise that the characteristic equation of A is $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$, where the roots for λ are e and f.

In our setting this equation becomes $\lambda^2 - t\lambda + u = 0$.

So $e + f = t = a + d$, and we need to find in our list of numerical values two pairs that add to one of the other numbers.

The only possibility is 4 + 14 = -13 + 31 = 18.

We also have $ef = u$, and $4 \times 14 = 56$ is not a number in our list.

Thus $e = -13, f = 31, a = 4, d = 14$, and $u = -13 \times 31 = -403$.

Thus $ad - bc = -403$, and $bc = 459$. Now 459 is in our list, so $b = 9, c = 51$ is possible, and nothing else works. Thus we have the matrix $\begin{pmatrix} 4 & 9 \\ 51 & 14 \end{pmatrix}$.

What are the eigenvectors here? Taking $e = -13$, we have

$$\begin{pmatrix} 4 & 9 \\ 51 & 14 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = -13 \begin{pmatrix} p \\ q \end{pmatrix}.$$

Thus $4p + 9q = -13p$, and $9q = -17p$. Thus all eigenvectors for this eigenvalue are of the form $\begin{pmatrix} 9p \\ -17p \end{pmatrix}$.

Check; $51p + 14q = -13q \Rightarrow 9q = -17p$.

Taking $f = 31$, we have

$$\begin{pmatrix} 4 & 9 \\ 51 & 14 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = 31 \begin{pmatrix} r \\ s \end{pmatrix}.$$

Thus $4r + 9s = 31r$, and $s = 3r$. Thus all eigenvectors for this eigenvalue are of the form $\begin{pmatrix} r \\ 3r \end{pmatrix}$.

Check; $51r + 14s = 31s \Rightarrow s = 3r$.

Looking at the numbers we have left, the only possibilities for $\begin{pmatrix} p \\ q \end{pmatrix}$ and $\begin{pmatrix} r \\ s \end{pmatrix}$ are $\begin{pmatrix} 45 \\ -85 \end{pmatrix}$ and $\begin{pmatrix} 30 \\ 90 \end{pmatrix}$.

Level 3

We need to find this time two triplets that add to one of the other numbers, so $1 + b + f = g + h + i = t$.

We have -6 + 16 + 11 = 21, but there is no second triplet to go with it.

Our only possible values are $1 + -6 + 16 = 1 + 3 + 7 = 11 = t$.

We must have $g = 1, h = 3, i = 7$ (since $1 \times -6 \times 16 = -96$ is not in our list of numbers) which means $b = -6, f = 16$ (since $b < f$). The eigenvalues must multiply to the determinant here, so u is 21.

Taking the top row of the matrix multiplied by the eigenvector $\begin{pmatrix} 0 \\ k \\ l \end{pmatrix}$ for the eigenvalue 1, we find that $k + l = 0$, and so $k = -l$. The only possible numbers are -51 and 51, and since $k < l, k = -51, l = 51$.

Multiplying the eigenvector now by the second row of the matrix gives $c = -7$, and now the third row gives $e = 15$.

Taking the top row of the matrix multiplied by the eigenvector $\begin{pmatrix} m \\ n \\ p \end{pmatrix}$ for the eigenvalue 3, we find that $m+n+p = 3m$, and so $n+p = 2m$. We know

$p = 2680$, so the only possible numbers are $m = 134, n = -2412$.

Taking the top row of the matrix multiplied by the eigenvector $\begin{pmatrix} q \\ r \\ s \end{pmatrix}$ for the eigenvalue 7, we find that $q + r + s = 7q$, and so $r + s = 6q$. The only possible numbers are $q = 270, r = -2880, s = 4500$, since $r < s$.

Now considering one of these last two eigenvectors multiplied by the second and third rows of the matrix gives $a = -22, d = 10$.

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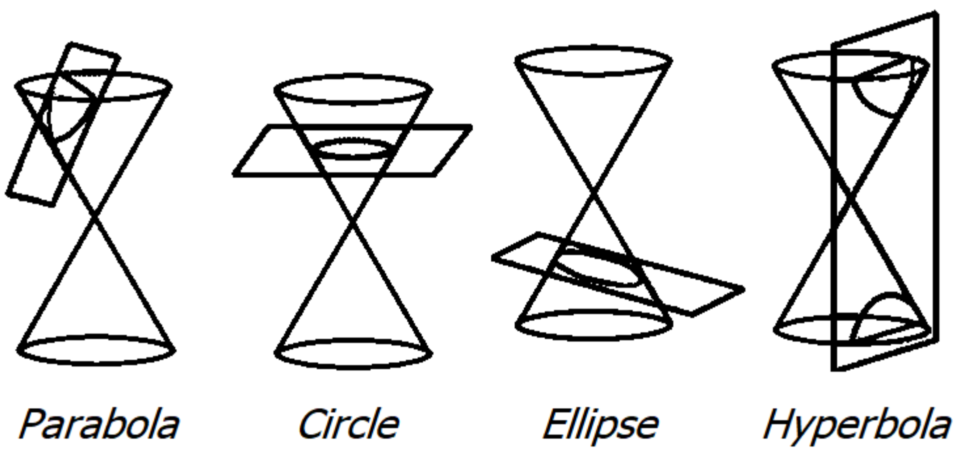
Further Risp 24: One-Parameter Conics: Notes

Topic: conics
Type of task: introductory
Preliminary knowledge required: none
Computing needs: graph plotting package

Level 1

Cutting a double cone yields these curves;



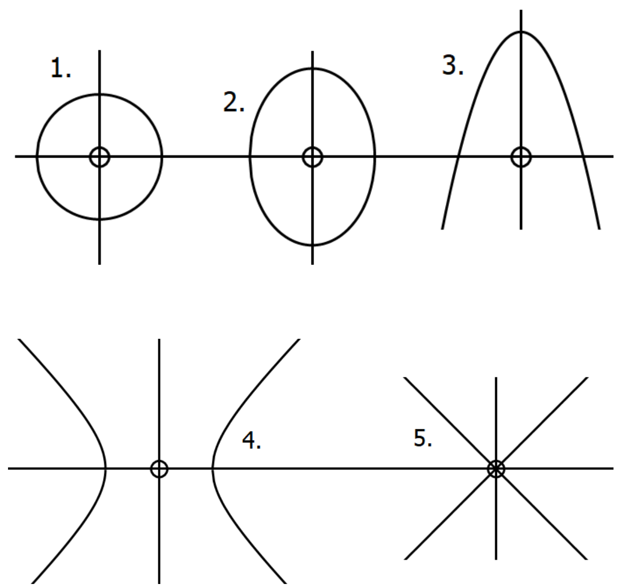
You can additionally make an intersecting pair of straight lines, if your cut goes through the vertex of the cone at the appropriate angle.

The focus-directrix-eccentricity definition also yields exactly the same collection of curves.

Every conic (as defined above) is a curve of the type

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

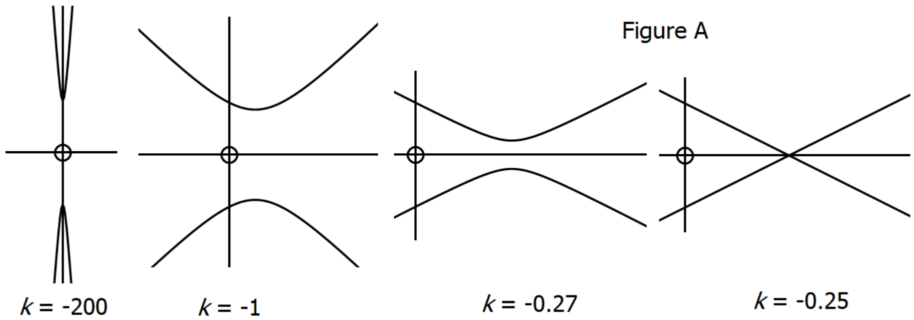
and every curve of this type is a conic.



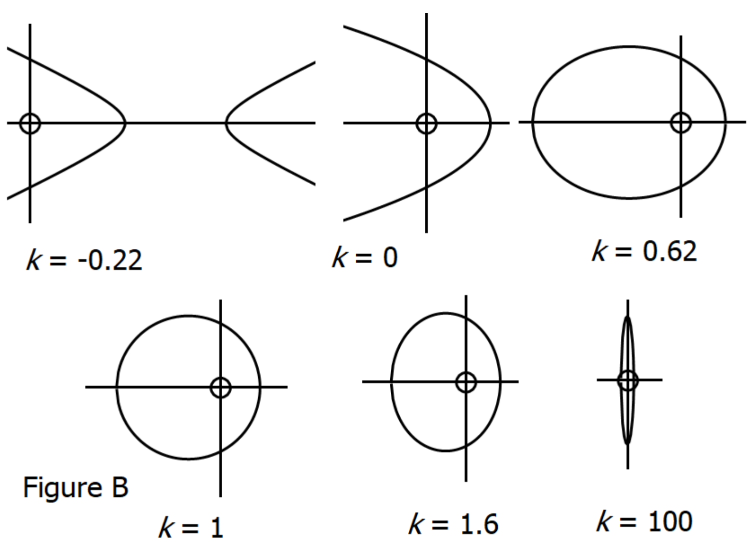
The eccentricity e varies as follows; for

1. the circle, $e = 0$,
2. the ellipse, $0 < e < 1$,
3. the parabola, $e = 1$,
4. the hyperbola, $e > 1$,
5. an intersecting pair of straight lines, $e = \infty$.

Level 2



We'll start with k large and negative (see Figure A) and gradually increase k. Initially we have a 'thin' hyperbola (where the acute angle between the asymptotes is small). As k increases, this angle increases, until at $k = -1$ the asymptotes are perpendicular (we have a *rectangular hyperbola*). If we increase k again, the hyperbola becomes a pair of straight lines when $k = -0.25$.



Increase k further (see Figure B), and we have a hyperbola again, until $k = 0$, when the curve becomes a parabola. As k becomes positive, we get a 'thin' ellipse, that gets 'wider' (more circular) as k increases.

Eventually it becomes a circle when $k = 1$. Beyond that, we have an ellipse again, that becomes 'thinner' as k increases.

Level 3

So which conics have we made? Our equation is of degree 2, so will always be a conic, but does it give all possible conics?

There is only one parabola and one circle to be made, and we have made these (all circles are similar, and all parabolas are similar).

We've made one pair of straight lines, containing an acute angle of $2\arctan 0.5 \approx 53.13°$.

We've made all possible ellipses, twice.

We've made all possible hyperbolae for $0 < \theta < 53.13°$ twice, and we have made all possible hyperbolae with $53.13° < \theta < 126.87°$ once, where θ is the smaller angle between the asymptotes.

But for any hyperbola where θ is greater than this, we fail.

Level 4

The standard equation for a central ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where a is the semi-major axis, and b is the semi-minor axis, and $a > b$. In this case $e = \sqrt{1 - \dfrac{b^2}{a^2}}$.

The standard equation for a central hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and in this case $e = \sqrt{1 + \dfrac{b^2}{a^2}}$.

Our curve $kx^2 + y^2 + x = 1$ is certainly a parabola for $k = 0$; we can write this for $k \neq 0$ as

$$\left(x + \frac{1}{2k}\right)^2 + \frac{y^2}{k} = \frac{1+4k}{4k^2}.$$

Say $X = x + \dfrac{1}{2k}, Y = y$, and we arrive (by translating the curve through $\dfrac{1}{2k}$ in the negative x-direction) at the curve

$$X^2 + \frac{Y^2}{k} = \frac{1+4k}{4k^2} \Rightarrow \frac{b^2}{a^2} = k.$$

If $k = 1$, we certainly have a circle, centre $(0, 0)$, radius $\sqrt{1.25}$.

If $0 < k < 1$, then the eccentricity of the ellipse is $\sqrt{1 - \dfrac{b^2}{a^2}} = \sqrt{1-k}$, and this takes all values from 0 to 1 as k takes all values between 0 and 1.

If $k > 1$, then the eccentricity of the ellipse is $\sqrt{1 - \dfrac{1}{k}} = \dfrac{\sqrt{k-1}}{\sqrt{k}}$, and this takes all values from 0 to 1 as k takes all values between 1 and ∞. So every ellipse appears twice as we vary k.

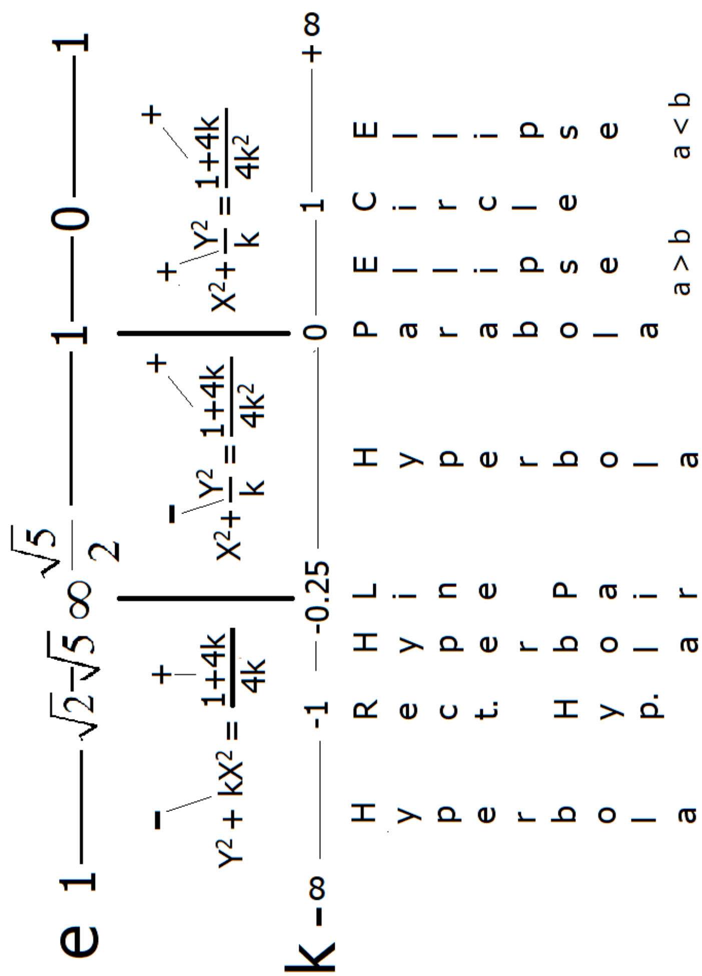
It's a different story for $k < 0$. For $-0.25 < k < 0$, we have a hyperbola where $e = \sqrt{1 + \dfrac{b^2}{a^2}} = \sqrt{1-k}$, so e can take all values between 1 and $\dfrac{\sqrt{5}}{2}$.

If $k = -0.25$, our equation becomes $X^2 - \dfrac{Y^2}{4} = 0$, so we have the two straight lines $Y = \dfrac{X}{2}$ and $Y = -\dfrac{X}{2}$.

For $k < -0.25$, e can take any value from $\sqrt{5}$ to 1.

So this is our final list; we've made a circle, a parabola, and every ellipse twice. We have made just one pair of straight lines, with gradients 0.5 and -0.5. We have made all hyperbolae with eccentricity between 1 and $\dfrac{\sqrt{5}}{2}$ twice, and each hyperbola with eccentricity in the range $\dfrac{\sqrt{5}}{2}$ to $\sqrt{5}$ once.

These results are summarised in the diagram below;



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Further Risp 25: Trace Arithmetic: Notes

Topic: matrices
Type of task: introductory
Preliminary knowledge required: basic matrix terminology
Computing needs: calculator for 3×3 matrices

Level 1

We have that $AP = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$.

So $\mathrm{tr}(AP) = ap + br + cq + ds$.

But $\mathrm{tr}(PA) = ap + br + cq + ds$ also, and so $\mathrm{tr}(AP) = \mathrm{tr}(PA)$, for any 2×2 matrices A and P.

Since we have used the general case here, this constitutes a proof.

Level 2

Checking particular cases here suggests that the same rule applies for 3×3 matrices.

We can prove this using subscripts. Let's say that A_{ij} is the element of A in the i^{th} row and the j^{th} column.

We can then say that $(AB)_{ij} = \sum_k A_{ik}B_{kj}$ as k runs through all possible values (we are adding the products of the elements in the i^{th} row of A with the j^{th} column of B). Now

$$\mathrm{tr}(AB) = \sum_i (AB)_{ii} = \sum_i \sum_j A_{ij}B_{ji} = \sum_j \sum_i B_{ji}A_{ij} = \sum_j (BA)_{jj} = \mathrm{tr}(BA).$$

This argument works for square matrices of any order.

Level 3

Say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $A^H = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$. We have $AA^H = \begin{pmatrix} ac+ba & ad+b^2 \\ c^2+ad & cd+bd \end{pmatrix}$.

So $\operatorname{tr}(A) = a+d, \operatorname{tr}(A^H) = c+b$, and

$$\operatorname{tr}(AA^H) = ac+ba+cd+bd = (a+d)(c+b) = \operatorname{tr}(A)\operatorname{tr}(A^H).$$

Since we have examined the general case here, this is a proof of the result that $\operatorname{tr}(AA^H) = \operatorname{tr}(A^H A) = \operatorname{tr}(A)\operatorname{tr}(A^H)$ for all 2×2 matrices A. We have the same result for A^V, that $\operatorname{tr}(AA^V) = \operatorname{tr}(A)\operatorname{tr}(A^V)$ for all 2×2 matrices A.

Does $\operatorname{tr}(AB) = \operatorname{tr}(A)\operatorname{tr}(B)$ imply that B is of the form A^H or A^V? We have again that $AP = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$.

So we need $(a+d)(p+s) = ap+br+cq+ds \Leftrightarrow as+dp = br+cq$. It is easy to find solutions to this equation, say

$$a=1, b=1, c=1, d=5, p=4, q=11, r=11, s=2.$$

These values yield that $\operatorname{tr}(AP) = \operatorname{tr}(A)\operatorname{tr}(P)$, and P is neither A^H or A^V.

Level 4

Intuition might lead us to guess that the rule $\operatorname{tr}(AA^H) = \operatorname{tr}(A)\operatorname{tr}(A^H)$ holds for square matrices of order higher than 2, but we would be wrong. Checking 3×3 matices on a calculator swiftly provides counterexamples to this idea. But we do have (with the help of a computer algebra package) that if $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $\operatorname{tr}(A)\operatorname{tr}(A^H) = \operatorname{tr}(AA^H)$ then $e = \dfrac{(b+h)(d+f)}{a+c+g+i}$.

The element e is the central one for A, and therefore there's a symmetry to this result, with almost a magic square feel to it. We get an exactly similar result for $A^V A$.

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Further Risp 26: The Coefficient Function: Notes

Topic: Maclaurin expansion, number theory
Type of task: consolidation
Preliminary knowledge required: expansions for common functions
Computing needs: none

Level 1

We know that $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}...$

and that $g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}....$

So we have that $f_3(x) = \frac{1}{6}, g_4(x) = -\frac{1}{4}$.

Level 2

The equation $f_3(px) + g_4(qx) = 0$ yields the equation

$$-\frac{p^3}{6} = -\frac{q^4}{4} \Rightarrow 2p^3 = 3q^4.$$

In finding positive solutions for p and q, it's the powers of 2 and 3 that are going to be important here.

Let $p = 2^m 3^n, q = 2^r 3^s$, where $m, n, r, s \geq 0$.

Thus on substituting in we get $2^{3m+1}3^{3n} = 2^{4r}3^{4s+1}$.

So we need $3m + 1 = 4r, 3n = 4s + 1$, and the smallest solutions here are $m = 1, n = 3, r = 1, s = 2$.

These yield $p = 54, q = 18$. The expansion for e^x is valid for all x, but the expansion for $\ln(1 + x)$ requires that $|x| < 1$.

So in this case, we need $|x| < \frac{1}{18}$.

Level 3

Are there any solutions to $ph_r(qx) = qi_r(px)$? The function h is cos, so its expansion contains only the even powers of x, while the function i is sin, so its expansion contains only the odd powers of x. Thus this equation has no solutions.

What about the equation $-5ph_q(rx) = rg_p((q-2)x)$? Substituting in carefully we arrive at

$$5(3)\frac{r^6}{6!} = r\frac{4^3}{4}.$$

Solving here for r gives $r = 4$.

Level 4

We are faced with solving $|2g_r(qx)| = |qh_2(rx)|$.

Let's first say $r = 2k+1$, where $k \geq 0$ is an integer, which gives us the equation

$$2\frac{q^{2k+1}}{2k+1} = q\frac{(2k+1)^2}{2!}$$

.

This gives us $4q^{2k} = (2k+1)^3$, and the left hand side is even while the right-hand side is odd, a contradiction.

So let's say instead that $r = 2k$, which gives us the equation

$$2\frac{q^{2k}}{2k} = q\frac{(2k)^2}{2!}$$

.

This gives us $q^{2k-1} = 2k^3$. Once again we can consider the powers of 2 here. Let $q = 2^m s, k = 2^n t$, where $s \geq 1$ and $t \geq 1$ are odd and $m, n \geq 0$.

Thus we have, after comparing the powers of two on each side,

$$(2(2^n t) - 1)m = 3n + 1 \Rightarrow m = \frac{3n+1}{2^{n+1}t - 1}.$$

Now m has to be an integer, but the denominator here is clearly going to grow much faster than the numerator as n increases. So we only need to try out small values of n and see what happens.

For $n = 0, m = \dfrac{1}{2t-1}$, and we have a solution when

$$t = 1, m = 1, k = 1, s = 1, r = 2, q = 2.$$

For $n = 1, m = \dfrac{4}{4t-1}$, and we have no solutions.

For $n = 2, m = \dfrac{7}{8t-1}$, we have a solution when

$$t = 1, m = 1, k = 4, s = 1, r = 8, q = 2.$$

For $n = 3, m = \dfrac{10}{16t-1} < 1$, so we have no solutions; we can see that already the denominator has outstripped the numerator, and there will be no further integer solutions for m.

Thus we have just the two solutions;

$$q = r = 2 \quad \text{and} \quad q = 2, r = 8.$$

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Further Risp 27: Hyperbolic Trigonometry: Notes

Topic: hyperbolic functions
Type of task: consolidation
Preliminary knowledge required: hyperbolic differentiation, trig identities
Computing needs: a graphing package

Level 1

The functions cos and cosh are even, while the functions sin and sinh are odd.

If you add or multiply two even functions, or multiply two odd functions, you get an even function, while if you add two odd functions, you get an odd function (why?).

So $C_1(x), C_2(x)$ and $S_1(x)$ are even, and $S_2(x)$ is odd.

Level 2

Let's write $\cos x$ as c, $\cosh x$ as $C, \sin x$ as s and $\sinh x$ as S. We will also leave out (x) from $C_1(x)$ and the other functions for simplicity.

We have $\dfrac{d}{dx}S_2 = C_2$. Also

$$\begin{aligned}\frac{d}{dx}S_1 &= cS + sC\\ &= (c+C)(S+s) - (CS + sc)\\ &= C_2S_2 - \frac{1}{2}(\sinh(2x) + \sin(2x))\\ &= C_2S_2 - \frac{1}{2}S_2(2x).\end{aligned}$$

Additionally,

$$\begin{aligned}
S_1(2x) &= \sin(2x)\sinh(2x) \\
&= 2sc \times 2SC \\
&= 4sScC \\
&= 4C_1S_1.
\end{aligned}$$

Also we have

$$\begin{aligned}
C_1(2x) &= \cos(2x)\cosh(2x) \\
&= (2c^2-1)(2C^2-1) \\
&= 4c^2C^2 - 2c^2 - 2C^2 + 1 \\
&= 4C_1^2 - 2(c+C)^2 + 4cC + 1 \\
&= 4C_1^2 - 2C_2^2 + 4C_1 + 1 \\
&= (2C_1+1)^2 - 2C_2^2.
\end{aligned}$$

Alternatively we could say

$$\begin{aligned}
C_1(2x) &= \cos(2x)\cosh(2x) \\
&= (c^2-s^2)(C^2+S^2) \\
&= c^2C^2 - s^2S^2 + c^2S^2 - s^2C^2 \\
&= c^2C^2 - s^2S^2 + (1-s^2)S^2 - (1-c)^2C^2 \\
&= 2c^2C^2 - 2s^2S^2 + S^2 - C^2 \\
&= 2c^2C^2 - 2s^2S^2 - 1 \\
&= 2C_1^2 - 2S_1^2 - 1.
\end{aligned}$$

Level 3

Curves need to share a point and also share a tangent at that point if they are to touch.

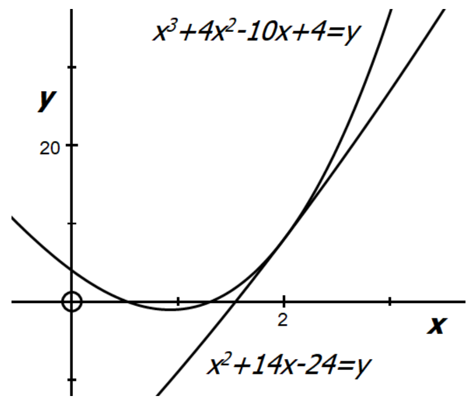
Consider, however, $y = x^3$ and $y = -x^3$; they share a point and a gradient when $x = 0$, but the curves cut each other here.

But as a general rule of thumb, these conditions for curves to touch work well.

If $x = 2$ then $x^3 + 4x^2 - 10x + 4$ is 8, and $x^2 + 14x - 10$ is 8 also.

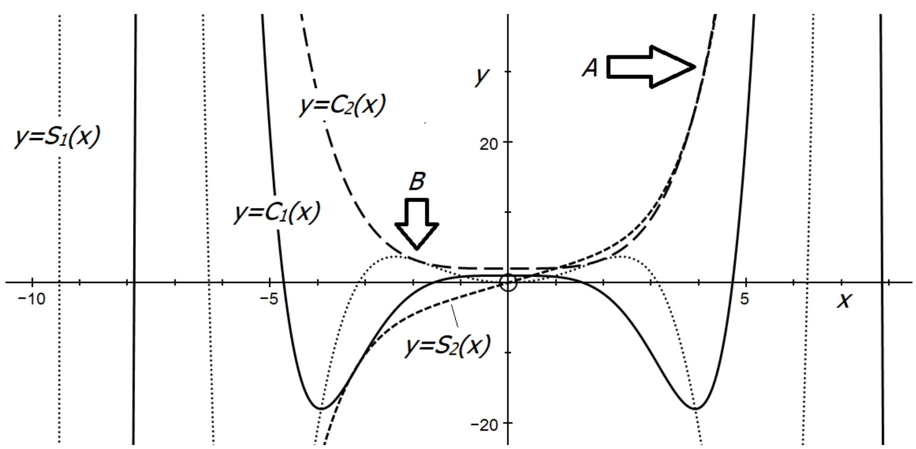
For the cubic curve we have $y' = 3x^2 + 8x - 10$, which is 18 at $x = 2$; the gradient of the parabola is $2x + 14$, which is 18 at $x = 2$.

Thus the two curves touch, as in this diagram;



Level 4

Drawing our four curves together on the same diagram gives this;



There appear to be possible touching points for $y = C_2$ and $y = S_2$ at A, for $y = S_1$ and $y = C_2$ at B (together with B's mirror image in the y-axis), while $y = C_1$ and $y = C_2$ could touch somewhere above A, as could $y = S_1$ and $y = C_1$.

We can check this last possibility by drawing the graph of $y = C_1 - S_1$. If there is a genuine touching point, then this graph will touch the x-axis at that x-value.

Looking at the four curves in combination carefully using this method, it seems that only B can be a genuine touching point, although there are some near misses.

Can we prove B is a touching point? From our Level 3 fact, we need for some value of x

1. $\sin x \sinh x = \cosh x + \cos x$, and (on differentiating)

2. $\cos x \sinh x + \sin x \cosh x = \sinh x + \sin x$.

Can these be true together? The first equation is true if and only if $c = sS - C$, which on substituting into the second equation gives

$$(sS - C)S + sC = S + s.$$

Making s the subject gives us $s = \dfrac{SC + S}{S^2 + C + 1} = \dfrac{S(C+1)}{C^2 + C} = \tanh x$.

Using $c = Ss - C$, we have $c = -\operatorname{sech} x$.

Now $s^2 + c^2 = \operatorname{sech}^2 x + \tanh^2 x = 1$.

Everything fits together nicely, and solutions exist here, found by solving $\sin x = \tanh x$. The first solution for positive x is $x = 1.875$ (4s.f.).

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Further Risp 28: Factorial Sums: Notes

Topic: telescoping sums
Type of task: introductory
Preliminary knowledge required: none
Computing needs: a computer algebra package is desirable

Level 1

The Wolfram Maths website tells us that

$$\sum_{r=1}^{n} r! = \frac{-e + Ei(1) + i\pi + E_{n+2}(-1)\Gamma(n+2)}{e},$$

where $Ei(x)$ is the exponential integral, $E_n(x)$ is the En-function, and $\Gamma(x)$ is the gamma function. These are all non-elementary integrals currently well beyond the A Level FM syllabus.

But $\sum_{r=1}^{n}[(r+1)! - r!]$ is straightforward. We have

$$\begin{aligned}\sum_{r=1}^{n}[(r+1)! - r!] &= (n+1)! + n! + (n-1)! + \ldots + 2! \\ &\quad - (n! + (n-1)! + \ldots + 2! + 1!) \\ &= (n+1)! - 1.\end{aligned}$$

Now $(r+1)! - r! = ((r+1) - 1)r! = r \times r! = f(r)r!$

So if $f(r) = r$ here, most terms cancel in the sum (we say the series 'telescopes') and finding $\sum_{r=1}^{n}[r \times r!]$ is easy.

Level 2

Are there any other choices for $f(r)$ where this happy cancelling takes place?

Choosing $f(r) = (r+1)$ doesn't work, because we are back to trying to find the sum of the factorial function again.

Choosing $f(r) = (r-1)$ doesn't work either, because we have here

$$(r-1)r! = ((r+1)-2)r! = (r+1)! - 2r!,$$

and we don't get enough cancelling when we sum here; the terms don't telescope neatly. We need all the factorials to cancel except for a few at the end and a few at the start.

Level 3

If $f(r) = ar + b$, then

$$(ar+b)r! = (a(r+1) + (b-a))r! = a(r+1)! + (b-a)r!$$

So the cancelling we want happens when $a + (b-a) = 0 \implies b = 0$ and a can be anything. For our successful Level 2 example above, $a = 1$.

What happens when we have a quadratic? Suppose we are trying to find

$$\sum_{r=1}^{n}(ar^2 + br + c)r!$$

Can we write $(ar^2 + br + c)r! = A(r+2)! + B(r+1)! + Cr!$?

We want the factorials to cancel on the right-hand side when we sum, so we must have $A + B + C = 0$.

Now the right hand side can be written as

$$r![A(r+2)(r+1) + B(r+1) + C] = r![Ar^2 + (3A+B)r + (2A+B+C)].$$

Comparing coefficients, we have $a = A, 3A + B = b$, and $2A + B + C = c$, which gives $A = a, B = b - 3a, C = c - b + a$.

Now we know $A+B+C = 0 \implies a+b-3a+c+a-b = 0 \implies c-a = 0.$

So if $c = a$, then b can be anything, and the series will telescope.

For example, if $a = 5, b = 11, c = 5$, then we need to find

$$\sum_{r=1}^{n}[(5r^2 + 11r + 5)r!].$$

Now $A = 5, B = -4, C = -1$, and the sum becomes

$$\begin{aligned}
&\sum_{r=1}^{n}[(5(r+2)(r+1) - 4(r+1) - 1)r!] \\
&\quad = \sum_{r=1}^{n} 5(r+2)! - 4(r+1)! - r! \\
&\quad = 5[(n+2)! + (n+1)! + n! + \ldots + 3!] \\
&\quad - 4[(n+1)! + n! + \ldots + 3! + 2!] - [n! + \ldots + 3! + 2! + 1!] \\
&\quad = 5[(n+2)! + (n+1)!] - 4[(n+1)! + 2!] - [2! + 1!] \\
&\quad = 5(n+2)! + (n+1)! - 11 \\
&\quad = (n+1)!(5n+11) - 11.
\end{aligned}$$

Level 4

What happens for the cubic? We have

$$\begin{aligned}
&(ar^3 + br^2 + cr + d)r! = A(r+3)! + B(r+2)! + C(r+1)! + Dr! \\
&= [A(r+3)(r+2)(r+1) + B(r+2)(r+1) + C(r+1) + D]r! \\
&\Rightarrow ar^3 + br^2 + cr + d \\
&= Ar^3 + r^2(6A + B) + r(11A + 3B + C) + 6A + 2B + C + D \\
&\Rightarrow a = A, B = b - 6a, C = c + 7a - 3b, D = -a + b - c.
\end{aligned}$$

Now we need $A + B + C + D = 0$, so

$a+b-6a+c+7a-3b-a+b-c = 0 \implies a-b+d = 0, c$ can be anything.

That's the cubic done; what happens for the quartic? The algebra is getting hard, so switching to a computer algebra system if we can will help us.

For the quartic, we find that we require $2a + b - c + e = 0$, while for the quintic, we find that $-9a + 2b + c - d + f = 0$ is the condition, and for the sextic, we discover we need $9a - 9b + 2c + d - e + g = 0$. The table below shows the coefficients of $a, b, c...$ as the length of the equation increases. A curious pattern seems to be emerging.

Degree	a	b	c	d	e	f	g
linear	0	1					
quadratic	-1	0	1				
cubic	1	-1	0	1			
quartic	2	1	-1	0	1		
quintic	-9	2	1	-1	0	1	
sextic	9	-9	2	1	-1	0	1

What is this sequence? The terms don't look obvious. In a situation like this, it's always a good idea to turn to the *Online Encyclopedia of Integer Sequences* at **www.oeis.org.** Typing in our first seven terms, that's 1, 0, -1, 1, 2, -9, 9, we get but a single exact match; $A014182$. This suggests our next term should be 50, and if we do the hard work for the degree 7 situation, 50 is indeed the coefficient that materialises.

So how is this sequence defined? If the Maclaurin series for $e^{1-x-e^{-x}}$ is $A_0 + A_1 x + A_2 \dfrac{x^2}{2!} + A_3 \dfrac{x^3}{3!} + \ldots$, then the sequence $A_0, A_1, A_2, A_3, \ldots$ is our sequence as in the table above (we say $e^{1-x-e^{-x}}$ is its *generating function*). How can this be? Is there any connection between our problem and this strange function? But the page devoted to $A014182$ shows lots of connections that seem equally outlandish. We know that the Maclaurin series for e^x involves the factorial function, so maybe there *is* a link in there somewhere!

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Further Risp 29: The Same Direction: Notes

Topic: complex numbers, loci in the complex plane
Type of task: consolidation
Preliminary knowledge required: basic complex number theory
Computing needs: a graphing package

Level 1

We have $(w+z)w = jz$, where j is real, and $zw + w = kz$, where k is real.

We can solve these simultaneously; the second equation gives $z = \dfrac{w}{k-w}$, and so from the first equation, $\left(w + \dfrac{w}{k-w}\right)w = j\dfrac{w}{k-w}$. Multiplying out and rearranging gives $w^2 - (k+1)w + j = 0$. Using the quadratic formula here, we have

$$w = \frac{k+1 \pm \sqrt{(k+1)^2 - 4j}}{2}.$$

Carrying out a similar procedure for z gives us

$$z = \frac{2j - k - k^2 \pm \sqrt{(k+1)^2 - 4j}}{2(k-j)}.$$

Note that if $(k+1)^2 \geq 4j$, both w and z are real, while if $(k+1)^2 < 4j$, both w and z have a non-zero imaginary part.

Level 2

Now $k = 1$ and $j > 1$, say $j = 1 + m^2$, where m is real and positive. Substituting in to our Level 1 work, we find

$$w = 1 \pm mi, z = -1 \pm \frac{1}{m}i.$$

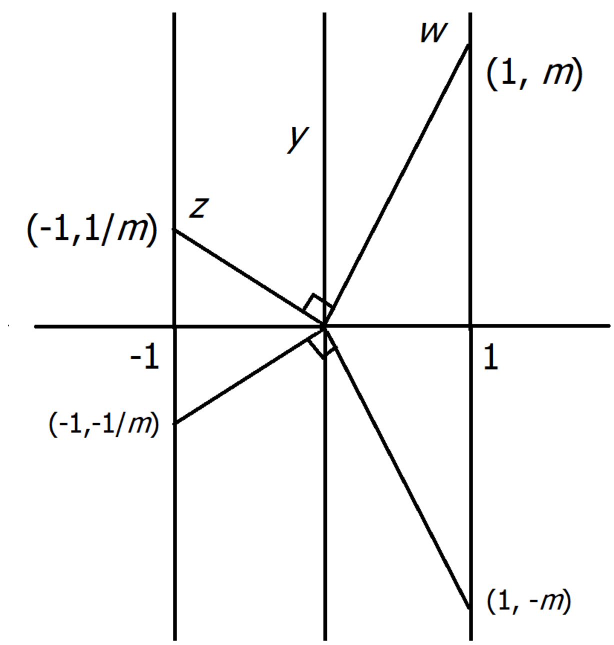
We must have here that $zw + w = z$, which means we have the two pairs, either

$$w = 1 + mi, z = -1 + \frac{1}{m}i$$

or

$$w = 1 - mi, z = -1 - \frac{1}{m}i.$$

So the angle between z and w is always $90°$. We could show this by examining gradients, or else we could notice that in the first case $z = \frac{iw}{m}$, while in the second $z = -\frac{iw}{m}$, and point out that multiplying by i or $-i$ has the effect of rotating a complex number through $90°$.



The minimum value of $|z - w|$ is clearly 2 when $m = 1$.

To confirm this, we have $|z - w| = \sqrt{2^2 + \left(m - \frac{1}{m}\right)^2}$, which is clearly a minimum when $m = 1$.

Level 3

Now for the second question, where $j = 1$ and $0 < k < 1$.

Substituting in we find

$$w = \frac{k+1}{2} \pm \frac{\sqrt{(k+3)(1-k)}i}{2}, z = -\frac{k+2}{2} \pm \frac{k\sqrt{k+3}}{2\sqrt{1-k}}i.$$

Given that $(w+z)w = z$, this yields the two pairs

$$w = \frac{k+1}{2} + \frac{\sqrt{(k+3)(1-k)}i}{2}, z = -\frac{k+2}{2} + \frac{k\sqrt{k+3}}{2\sqrt{1-k}}i$$

and

$$w = \frac{k+1}{2} - \frac{\sqrt{(k+3)(1-k)}i}{2}, z = -\frac{k+2}{2} - \frac{k\sqrt{k+3}}{2\sqrt{1-k}}i.$$

So looking at w for the first case, we have $x = \dfrac{k+1}{2} \Rightarrow k = 2x - 1$, and so $\dfrac{1}{2} < x < 1$.

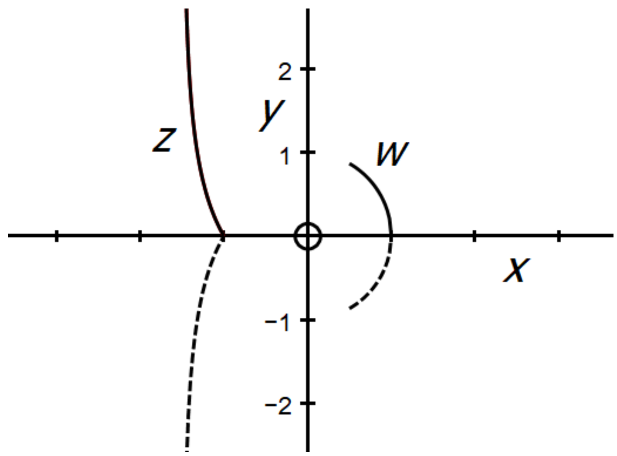
We also have $y = \dfrac{\sqrt{(k+3)(1-k)}}{2} \Rightarrow y = \sqrt{1-x^2}$.

Looking at z similarly, $x = -\dfrac{k+2}{2} \Rightarrow k = -2x - 2, -\dfrac{3}{2} < x < -1$.

Thus $y = \dfrac{k\sqrt{k+3}}{2\sqrt{1-k}} \Rightarrow y = -\dfrac{(x+1)\sqrt{1-2x}}{\sqrt{2x+3}}$.

A similar process for the second pair of equations gives $y = -\sqrt{1-x^2}$ for w, and $y = \dfrac{(x+1)\sqrt{1-2x}}{\sqrt{2x+3}}$ for z.

Plotting these curves together gives us this, with the first pair of equations solid and the second pair dashed;



There is an alternative approach here. Along which path does z travel? Assuming w travels in a circle centre the origin, we can say $w = \cos\phi + i\sin\phi$, where $0 < \phi < \dfrac{\pi}{3}$. Now

$$\begin{aligned}
z = \frac{w^2}{1-w} &= \frac{e^{2\phi i}}{1-\cos\phi - i\sin\phi} \\
&= \frac{e^{2\phi i}}{1-(1-2\sin^2\frac{\phi}{2}) - 2i\sin\frac{\phi}{2}\cos\frac{\phi}{2}} \\
&= \frac{e^{2\phi i}}{2\sin\frac{\phi}{2}(\sin\frac{\phi}{2} - i\cos\frac{\phi}{2})} \\
&= \frac{e^{2\phi i}}{-2i\sin\frac{\phi}{2}(i\sin\frac{\phi}{2} + \cos\frac{\phi}{2})} \\
&= \frac{e^{2\phi i}}{-2i\sin\frac{\phi}{2}(e^{i\frac{\phi}{2}})} \\
&= \frac{e^{i\frac{3\phi}{2}+i\frac{\pi}{2}}}{2\sin\frac{\phi}{2}}.
\end{aligned}$$

So if $z = re^{i\theta}$, then $r = \dfrac{1}{2\sin\frac{\phi}{2}}, \theta = \dfrac{3\phi}{2} + \dfrac{\pi}{2}$, and the equation of the polar curve for the locus of z is

$$r = \frac{1}{2\sin(\theta/3 - \pi/6)},$$

where since $0 < \phi < \dfrac{\pi}{3}$, we have $\dfrac{\pi}{2} < \theta < \pi$. We arrive at the same curve as above.

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Further Risp 30: A Matrix with Unit Elements: Notes

Topic: determinant, trace
Type of task: introductory
Preliminary knowledge required:
calculation of determinants for 2×2 and 3×3, Sarrus' method
Computing needs: none

Level 1

I suppose you might ask, 'Why is the trace important?' It is a footnote in the syllabus, for sure. But it is always the sum of the eigenvalues of a matrix, and thus makes an appearance in its characteristic equation.

		Trace	
Determinant	*Less*	*Same*	*More*
Less	$\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
Same	$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$
More	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

This is one solution. Notice that for 2×2 matrices, $\det(-A) = \det(A)$, and $\operatorname{tr}(-A) = -\operatorname{tr}(A)$, so the three matrices on the left 'reflect' in the central vertical line to get the three on the right via negation.

There are five possible values for the determinant, so a matrix in the top row must have determinant -1 or -2, a matrix in the second row must have determinant 0, and a matrix in the bottom row must have determinant 1 or 2. The same is true for the trace values.

Level 2

If we multiply every element of every matrix by k, the trace is enlarged by a scale factor of k each time, while the determinant is enlarged by a scale factor of k^2, and so the new grid remains truthful if $k = 2$. If $k = -2$, the determinants are multiplied by 4, while the traces are multiplied by -2, so we would have to adjust our table accordingly.

If, however, we add 1 onto every element of every matrix, the trace is increased by 2 each time, but the fate of the determinant is less predictable. For the answer above, the determinants of the three matrices in the middle row become -2, 0 and 2 respectively, so the table no longer holds.

Level 3

Determinant \ Trace	Less	Same	More
Less	$\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Same	$\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
More	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

This is one solution. Notice that for 3×3 matrices, $\det(-A) = -\det(A)$, and $\operatorname{tr}(-A) = -\operatorname{tr}(A)$, so really we only need to find four well-chosen matrices together with the central matrix, and then 'reflect' them in the central square to get the others via negation.

There are, in fact, only three possible values for the determinant, -1, 0 and 1, so a matrix in the top row must have determinant -1, a matrix in the second row must have determinant 0, and a matrix in the bottom row must have determinant 1. Are we sure that the determinant here could not be, say, 2? Let's consider Sarrus's rule for finding the determinant.

$$\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \\ a & b & c \\ d & e & f \end{array}$$

To get at least one non-zero contribution to the determinant, one diagonal line (say g–b–f in the above) must be all 1s or -1s. This means the other two diagonals with the same slant share five zeros, which means they will contribute nothing, and one of these diagonals must be all zeros. Looking at the diagonals going the other way, this means that they must all contain a zero (contributed by the all-zero diagonal going the other way), and so they will contribute nothing either. Thus the determinant of the matrix must be -1, 0 or 1.

Level 4

If we multiply every element of every matrix by k, the trace is enlarged by a scale factor of k, while the determinant is enlarged by a scale factor of k^3, and so the new grid remains truthful if $k = 2$. If $k =$ -2, then we need to reverse the order for **less–same–more** for both the trace and the determinant. If, however, we add 1 onto every element of every matrix, the trace is increased by 3 each time, but the fate of the determinant is less predictable. For the answer above, the determinants of the three matrices in the middle row become 3, -3 and 3 respectively, so the table no longer holds.

As a curiosity, you might note that

$$\begin{vmatrix} a+k & b+k & c+k \\ d+k & e+k & f+k \\ g+k & h+k & i+k \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} k & k & k \\ a & b & c \\ d & e & f \end{vmatrix} + \begin{vmatrix} k & k & k \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} k & k & k \\ g & h & i \\ a & b & c \end{vmatrix}.$$

This risp is on page 68; Home Page

Further Risp 31: Matrix Segments: Notes

Topic: linear transformations and matrices
Type of task: introductory
Preliminary knowledge required:
basic matrix and linear transformation theory
Computing needs: none

Level 1

Suppose $A = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $B = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$. We wish to map AB onto DE, where $C = \begin{pmatrix} x_4 \\ y_4 \end{pmatrix}$ and $D = \begin{pmatrix} x_5 \\ y_5 \end{pmatrix}$.

We're looking for a matrix $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ so that $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_4 \\ y_4 \end{pmatrix}$, and $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_5 \\ y_5 \end{pmatrix}$.

We can combine these to give $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_4 & x_5 \\ y_4 & y_5 \end{pmatrix}$.

If $x_1y_2 - x_2y_1 \neq 0$, then the inverse matrix of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ exists, and $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \dfrac{1}{x_1y_2 - x_2y_1} \begin{pmatrix} x_4 & x_5 \\ y_4 & y_5 \end{pmatrix} \begin{pmatrix} y_2 & -x_2 \\ -y_1 & x_1 \end{pmatrix}$.

When does $x_1y_2 - x_2y_1 = 0$? When $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2}$ (assuming y_1 and y_2 are non-zero), which is when the points A, B and the origin lie on a straight line (equivalently, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$). Note that $\begin{pmatrix} x_4 & x_5 \\ y_4 & y_5 \end{pmatrix}$ can be singular here (that would mean we couldn't map CD to AB with a matrix).

Level 2

We need to find p, q, r and s so that

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_3 + t & x_4 + t \\ y_3 + u & y_4 + u \end{pmatrix}.$$

As before, we can post-multiply by the inverse of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ to give (assuming $x_1 y_2 - x_2 y_1 \neq 0$)

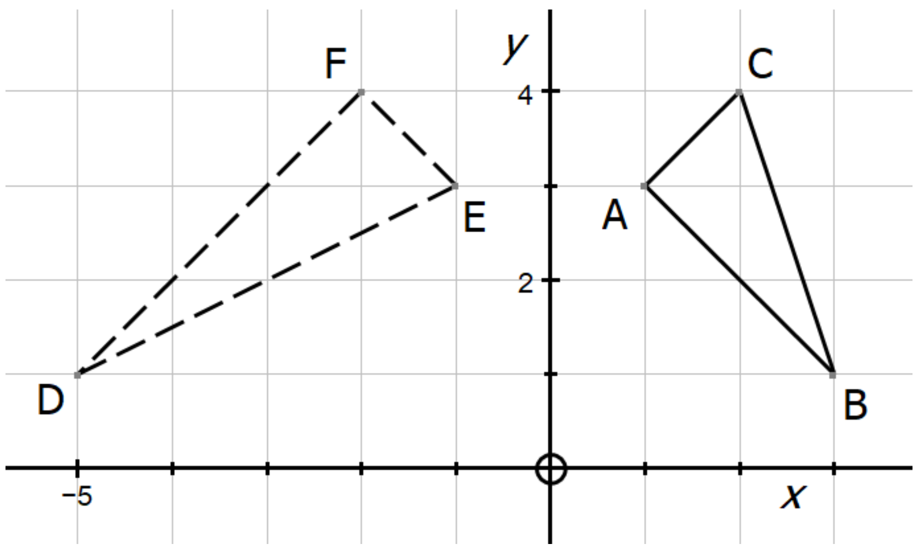
$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \frac{1}{x_1 y_2 - x_2 y_1} \begin{pmatrix} x_3 + t & x_4 + t \\ y_3 + u & y_4 + u \end{pmatrix} \begin{pmatrix} y_2 & -x_2 \\ -y_1 & x_1 \end{pmatrix}$$

Level 3

Can we map any triangle ABC to any triangle DEF with a matrix? Our work suggests that we can certainly map a side to a side, but if the third vertex then maps to the third vertex, we will have been fortunate indeed. But we get more than one chance. The line segments AB, BA, AC, CA, BC and CB could all map to DE, and we could replace DE here with ED, DF, FD, EF and FE. So that's 36 chances in total for the third vertex to land on the third vertex.

Level 4

Can we always map a triangle ABC to a triangle GHI that is *congruent* to a given triangle DEF (but which may be in a different part of the plane)? Let's take an example, and suppose that we can do even better, and map ABC onto a triangle that is a translation of DEF. In the diagram below, we will map AB to DE, and our translation will be through $\begin{pmatrix} t \\ u \end{pmatrix}$.

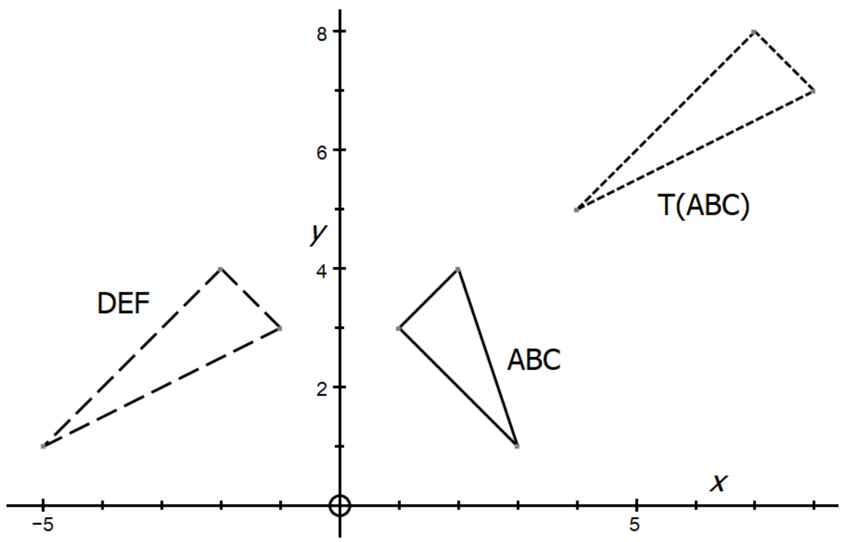


So we have that $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} t-5 & t-1 \\ u+1 & u+3 \end{pmatrix}$.

So $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = -\frac{1}{8} \begin{pmatrix} t-5 & t-1 \\ u+1 & u+3 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} \dfrac{t+1}{4} & \dfrac{t-7}{4} \\ \dfrac{u+4}{4} & \dfrac{u}{4} \end{pmatrix}$.

Now $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} t-2 \\ u+4 \end{pmatrix}$, which gives us (on substituting in for $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$), $\begin{pmatrix} \dfrac{3t-13}{2} \\ \dfrac{3u+4}{2} \end{pmatrix} = \begin{pmatrix} t-2 \\ u+4 \end{pmatrix}$.

These solve to give $t = 9, u = 4$, and if T is the transformation represented by $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 2.5 & 0.5 \\ 2 & 1 \end{pmatrix}$, then $T(ABC)$ is the triangle in the image below.



So it seems that in general, if we add the set of translations to the set of linear transformations (which can be represented by matrices), we can always (except for cases where a side of the triangle goes through the origin) map any triangle onto any other (a translation can't be represented by a matrix, and so is not a linear transformation).

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Further Risp 32: Binary Operation Venn Diagram: Notes

Topic: binary operations and groups
Type of task: introductory
Preliminary knowledge required: basic group definition
Computing needs: none

Anyone who takes a university maths course is likely to meet groups in their first year. They also appear in some Further Maths courses.

Level 1

I was once given a useful mnemonic for the groups axioms;

I: Inverses
K: Klosure (sorry!)
e: Identity
A: Associativity

These are group axioms you have to put together for yourself(!) Notice that commutativity is not part of a group's definition.

Level 2

$+$	$\mathbb{N}$	$\mathbb{Z}$
Associative	✓	✓
Commutative	✓	✓
Closed	✓	✓

$-$	$\mathbb{N}$	$\mathbb{Z}$
Associative	✗	✗
Commutative	✗	✗
Closed	✗	✓

Level 3

The great thing about trying to fill a Venn diagram like this one is that every example of a binary operation on the real numbers excluding zero that you come up with must live in one of these regions. This means that brainstorming without too much thought at the start is bound to generate some viable answers.

Let's try our easy operations first. The operation $x + y$ is associative, commutative but not closed, since $-2 + 2 = 0$, which is not in our set. So that's region 5 filled.

We have $x \times y$ is associative, commutative and closed, since multiplying two non-zero numbers cannot give zero, so region 8 is dealt with.

Also $x - y$ is not associative, or commutative, or closed, since $2 - 2 = 0$. So we have an example for region 1.

The operation x/y, however, is not associative or commutative, but it counts as closed, since dividing two non-zero numbers cannot give zero. Region 3 is now filled.

Now things get harder. How about $xoy = \dfrac{1}{xy}$? This is not associative, since $\dfrac{1}{\dfrac{1}{xy}z} \neq \dfrac{1}{x\dfrac{1}{yz}}$. But it is commutative, and also closed, so this fits into region 7.

We could try $xoy = \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{x+y}{xy}$. Commutative, certainly, but not closed, and associativity doesn't work either. So region 4 has now succumbed.

For regions 2 and 6, we can try a deceptively simple possibility. If we define $xoy = x$, then this proves to be associative and closed but not commutative (region 6 dealt with).

This suggests the further operation $xoy = x-1$, which is not commutative and not closed this time, since $1oy = 0$. But

$$(xoy)oz = x - 2, xo(yoz) = x - 1,$$

so we do not have associativity, and region 2 remains stubbornly difficult.

Thus examples have been found for seven regions of the Venn diagram; we are left needing to find an associative, non-commutative operation on the reals excluding zero where we do not have closure. An example will be received with open arms!

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Further Risp 33: Adding Three Integrals: Notes

Level 1

$\int \frac{1}{1+x^2}dx = \arctan x + k$ (use the substitution $x = \tan u$).

$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\frac{x}{a} + k.$

$\int \frac{b^2}{a^2+c^2x^2}dx = \frac{b^2}{ac}\arctan\frac{cx}{a} + k.$

Level 2

We could use the third result above, or writing it out in full;

$$\begin{aligned}
I_1 = \int_d^e \frac{b}{a+cx^2}dx &= \frac{b}{a}\int_d^e \frac{1}{1+\left(\sqrt{\frac{c}{a}}x\right)^2}dx \\
&= \frac{b}{a}\left[\frac{\arctan\left(\sqrt{\frac{c}{a}}x\right)}{\sqrt{\frac{c}{a}}}\right]_d^e \\
&= \frac{b}{a}\times\frac{\sqrt{a}}{\sqrt{c}}\left[\arctan\left(\sqrt{\frac{c}{a}}x\right)\right]_d^e \\
&= \frac{b}{\sqrt{ac}}(\arctan(re) - \arctan(rd)) \\
&= \arctan(re) - \arctan(rd).
\end{aligned}$$

Level 3

In exactly the same way, $\int_e^a \frac{c}{b+dx^2}dx = \arctan(ra) - \arctan(re)$, and $\int_a^b \frac{d}{c+ex^2}dx = \arctan(rb) - \arctan(ra)$.

Thus

$$\begin{aligned} I_1 + I_2 + I_3 &= \int_d^e \frac{b}{a+cx^2}dx + \int_e^a \frac{c}{b+dx^2}dx + \int_a^b \frac{d}{c+ex^2}dx \\ &= \arctan(rb) - \arctan(rd). \end{aligned}$$

Now if a, b, c, d and e are geometric progression with common ratio r, then a, c and e are also in geometric progression, with common ratio r^2.

Thus $\int_p^q \frac{c}{a+ex^2}dx = \arctan(r^2q) - \arctan(r^2p)$.

So if we choose $p = \frac{d}{r}$, and $q = \frac{b}{r}$, we have

$$\int_{d/r}^{b/r} \frac{c}{a+ex^2}dx = \int_d^e \frac{b}{a+cx^2}dx + \int_e^a \frac{c}{b+dx^2}dx + \int_a^b \frac{d}{c+ex^2}dx.$$

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Further Risp 34: The Series Result: Notes

Topic: telescoping sums
Type of task: introductory
Preliminary knowledge required: partial fractions
Computing needs: none

Level 1

Suppose we pick $a = 7$. To start, we need to find $S_{7,8,100} = \sum_{1}^{5} \frac{1}{(r+7)(r+8)}$.

We can split $\frac{1}{(r+7)(r+8)}$ into partial fractions, giving

$$\frac{1}{(r+7)(r+8)} = \frac{1}{r+7} - \frac{1}{r+8}.$$

Now we have

$$S_{7,8,100} = \left(\frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{106} + \frac{1}{107}\right) - \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{107} + \frac{1}{108}\right) = \frac{1}{8} - \frac{1}{13}.$$

Similarly we have $S_{8,9,100} = \frac{1}{9} - \frac{1}{109}$.

How about $S_{7,9,100}$? We can split $\frac{1}{(r+7)(r+9)}$ into partial fractions, giving $\frac{1}{(r+7)(r+9)} = \frac{1}{2}\left(\frac{1}{r+7} - \frac{1}{r+9}\right)$.

Cancelling fractions as before, we find $S_{7,9,100} = \frac{1}{2}\left(\frac{1}{8} + \frac{1}{9} - \frac{1}{108} - \frac{1}{109}\right)$, and thus $\frac{S_{7,8,100} + S_{8,9,100}}{S_{7,9,100}} = 2$.

Level 2

Our conjecture is that $\frac{S_{a,a+1,n} + S_{a+1,a+2,n}}{S_{a,a+2,n}} = 2$ for all positive integer a and n.

It is easy to extend the argument we constructed for Level 1 for the general case.

Level 3

Consider the case where we choose the numbers a, $a+d = b$, and $a+2d = c$, where d is a positive integer; does our result still hold? We have

$$S_{a,a+d,n} =$$

$$\frac{1}{d}\left[\left(\frac{1}{1+a}+\frac{1}{2+a}+\cdots+\frac{1}{n+a}\right)-\left(\frac{1}{1+a+d}+\cdots+\frac{1}{n+a+d}\right)\right],$$

where some fractions may well cancel. Also,

$$S_{a+d,a+2d,n} =$$

$$\frac{1}{d}\left[\left(\frac{1}{1+a+d}+\frac{1}{2+a+d}+\cdots+\frac{1}{n+a+d}\right)-\left(\frac{1}{1+a+2d}+\cdots+\frac{1}{n+a+2d}\right)\right],$$

where some fractions may well cancel. In addition,

$$S_{a,a+2d,n} =$$

$$\frac{1}{2d}\left[\left(\frac{1}{1+a}+\frac{1}{2+a}+\cdots+\frac{1}{n+a}\right)-\left(\frac{1}{1+a+2d}+\frac{1}{2+a+2d}+\cdots+\frac{1}{n+a+2d}\right)\right].$$

Thus it is clear that $\dfrac{S_{a,a+d,n}+S_{a+d,a+2d,n}}{S_{a,a+2d,n}} = 2$. The value for a does not need to be a whole number here, and neither does the value for d.

Level 4

Consider the case where $a < b < c$ can be any three positive whole numbers that are *not* consecutive terms from an arithmetic progression. Does the result $\dfrac{S_{a,b,n} + S_{b,c,n}}{S_{a,c,n}} = 2$ still hold?

Let's check a particular case; is it true that $\dfrac{S_{5,7,100} + S_{7,10,100}}{S_{5,10,100}} = 2$? The answer is 'No'; evaluating this fraction yields 2.071(4s.f.)

The key problem concerns the fraction at the front of our fraction sum, which is $\dfrac{1}{d}$ in the Level 3 argument. If $b - a \neq c - b$, then we get a different fraction at the front of our sum of fractions in the numerator each time, which means our argument breaks down.

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Further Risp 35: The k-matrix: Notes

Topic: matrices (and possibly some groups)
Type of task: consolidation
Preliminary knowledge required:
eigenvalues and eigenvectors, group definitions
Computing needs: none

Level 1

Let's take a concrete example to start with. Suppose we pick the numbers $a = 3, b = 4, c = 5, d = 6$, and $k = 7$.

Thus $A = \begin{pmatrix} 3 & 4 \\ 28 & 3 \end{pmatrix}, B = \begin{pmatrix} 5 & 6 \\ 42 & 5 \end{pmatrix}$.

Adding, we have $A+B = \begin{pmatrix} 8 & 10 \\ 70 & 8 \end{pmatrix} \Rightarrow A, B$ and $A+B$ are all 7-matrices.

The eigenvalues of A are given by solving $\det(A - \lambda I) = 0$, and are

$$\lambda_1 = 3 - 4\sqrt{7}, \lambda_2 = 3 + 4\sqrt{7}.$$

The eigenvalues of B are

$$\lambda_3 = 5 - 6\sqrt{7}, \lambda_4 = 5 + 6\sqrt{7},$$

while the eigenvalues of $A + B$ are

$$\lambda_5 = 8 - 10\sqrt{7}, \lambda_6 = x = 8 + 10\sqrt{7}.$$

So the eigenvalues of $A + B$, ordered by size, are the sums of the eigenvalues of A and B, ordered by size, that is, $\lambda_1 + \lambda_3 = \lambda_5, \lambda_2 + \lambda_4 = \lambda_6$.

When we multiply, we find $AB = \begin{pmatrix} 183 & 38 \\ 266 & 183 \end{pmatrix}$.

Now the eigenvalues of AB are $\lambda_7 = 183 - 38\sqrt{7}, \lambda_8 = 183 + 38\sqrt{7}$, and

$$(3 - 4\sqrt{7})(5 - 6\sqrt{7}) = 183 - 38\sqrt{7}, (4\sqrt{7} + 3)(6\sqrt{7} + 5) = 183 + 38\sqrt{7}.$$

Thus the eigenvalues of AB, ordered by size, are the eigenvalues of A and B, ordered by size, multiplied together, that is, $\lambda_1\lambda_3 = \lambda_7, \lambda_2\lambda_4 = \lambda_8$.

Now let's tackle the general case. We have $A = \begin{pmatrix} a & b \\ kb & a \end{pmatrix}, B = \begin{pmatrix} c & d \\ kd & c \end{pmatrix}$.

Adding, we find $A + B = \begin{pmatrix} a+c & b+d \\ k(b+d) & a+c \end{pmatrix}$, which is also a k-matrix.

Note the value of k is unchanged here. If k changed, we could not call it a k-matrix!

Multiplying, we find $AB = \begin{pmatrix} ac+bdk & ad+bc \\ k(ad+bc) & ac+bdk \end{pmatrix}$, which is also a k-matrix.

Level 2

We can find the eigenvalues of $\begin{pmatrix} a & b \\ kb & a \end{pmatrix}$ by solving the equation

$$\det \begin{pmatrix} a-\lambda & b \\ kb & a-\lambda \end{pmatrix} = 0.$$

This gives us $(a-\lambda)^2 - kb^2 = 0$, which gives us that

$$\lambda_1 = a - \sqrt{k}b, \lambda_2 = a + \sqrt{k}b, \lambda_1 \leq \lambda_2,$$

since b and k are non-negative. Looking now at $B = \begin{pmatrix} c & d \\ kd & c \end{pmatrix}$, we have

$\lambda_3 = c - \sqrt{k}d, \lambda_4 = c + \sqrt{k}d$ with $\lambda_3 \leq \lambda_4$ since d and k are non-negative.

Looking now at $A + B = \begin{pmatrix} a+c & b+d \\ k(b+d) & a+c \end{pmatrix}$, we have

$$\lambda_5 = a + c - \sqrt{k}(b+d), \lambda_6 = a + c + \sqrt{k}(b+d)$$

with $\lambda_5 \leq \lambda_6$ since $b + d$ and k are non-negative.

We can see that $\lambda_1 + \lambda_3 = \lambda_5, \lambda_2 + \lambda_4 = \lambda_6$.

Looking now at $AB = \begin{pmatrix} ac+bdk & ad+bc \\ k(ad+bc) & ac+bdk \end{pmatrix}$, we have

$$\lambda_7 = ac + bdk - \sqrt{k}(ad+bc), \lambda_8 = ac + bdk + \sqrt{k}(ad+bc)$$

with $\lambda_5 \leq \lambda_6$ since $ad + bc$ and k are non-negative.

This is harder to spot this time, but $\lambda_1\lambda_3 = \lambda_7, \lambda_2\lambda_4 = \lambda_8$.

Are there any other 2×2 matrices P, Q where the eigenvalues of P and Q add to give the eigenvalues of $P + Q$, and multiply to give the eigenvalues of PQ? Time spent playing with a computer algebra package yields this example;

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix}, A + B = \begin{pmatrix} 5 & 7 \\ 9 & 7 \end{pmatrix}, AB = \begin{pmatrix} 16 & 15 \\ 24 & 25 \end{pmatrix}.$$

$$\lambda_1 = -1, \lambda_2 = 4, \lambda_3 = -1, \lambda_4 = 10, \lambda_5 = -2, \lambda_6 = 14, \lambda_7 = 1, \lambda_8 = 40.$$

So a pair of matrices here that are not k-matrices for any k, where their eigenvalues add to the eigenvalues of their sum, and their eigenvalues multiply to the eigenvalues of their product.

There does seem to be a theorem of sorts waiting to be uncovered here. Here's a conjecture;

The eigenvalues (ordered by size) of the 2×2 matrices A and B add to the eigenvalues of $A + B$ (ordered by size) if and only if they multiply to the eigenvalues of AB (ordered by size).

This is left for the reader to explore!

Level 3

If $A = \begin{pmatrix} a & b \\ kb & a \end{pmatrix}$ then we know the eigenvalues of A are $a \pm \sqrt{k}b$.

Thus the eigenvectors of A are given by $\begin{pmatrix} \pm\sqrt{k}b & b \\ kb & \pm\sqrt{k}b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which solve to give $\begin{pmatrix} 1 \\ -\sqrt{k} \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{k} \end{pmatrix}$. These are independent of a and b.

Adding them gives $\begin{pmatrix}2\\0\end{pmatrix}$, which is independent of k as well. Their scalar product is $1-k$, so the eigenvectors are perpendicular if and only if $k=1$.

Level 4

Let's ease the restriction now that a and b need to be non-negative.

We've seen that two k-matrices add to a k-matrix, and that two k-matrices multiply to a k-matrix.

The additive identity is $\begin{pmatrix}0&0\\0&0\end{pmatrix}$, which is a k-matrix for any k.

The multiplicative identity is $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, which is again a k-matrix for any k.

Matrix multiplication and addition are always associative.

Do we have inverses? We can see $\begin{pmatrix}-a&-b\\-bk&-a\end{pmatrix}$ is the additive inverse for A, which is a k-matrix.

We can see that $\dfrac{1}{a^2-kb^2}\begin{pmatrix}a&-b\\-bk&a\end{pmatrix}$ is the multiplicative inverse for A, which is a k-matrix. There is a problem, however, if $a^2-kb^2=0$, when the k-matrix is singular.

We are edging towards wondering if non-singular k-matrices might form a field here, that is, an abelian group with respect to addition, and (excluding the zero element) an abelian group with respect to multiplication. We have identities, inverses, associativity, and closure; we do have abelian additive and multiplicative groups, and multiplication distributes across addition, so we have a field. Note that although matrix multiplication in general is not commutative, multiplying two k-matrices together is commutative.

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Further Risp 36: The Cycling Lines: Notes

Topic: vectors
Type of task: revision
Preliminary knowledge required: general vector theory
Computing needs: none

Level 1

We end up with three lines, let's call them Line 1, Line 2 and Line 3;

$$\underline{r} = \begin{pmatrix} 1 \\ a \\ 2a^2 \end{pmatrix} + \lambda \begin{pmatrix} a^2 \\ 2 \\ 2a \end{pmatrix}, \underline{r} = \begin{pmatrix} 3 \\ 3a \\ 4a^2 \end{pmatrix} + \mu \begin{pmatrix} 3a^2 \\ 4 \\ 4a \end{pmatrix}, \underline{r} = \begin{pmatrix} 5 \\ 5a \\ 6a^2 \end{pmatrix} + \delta \begin{pmatrix} 5a^2 \\ 6 \\ 6a \end{pmatrix}.$$

$$\begin{vmatrix} 1 & a^2 & 3a^2 \\ a & 2 & 4 \\ 2a^2 & 2a & 4a \end{vmatrix} = \begin{vmatrix} 3 & a^2 & 3a^2 \\ 3a & a & 4 \\ 4a^2 & 2a & 4a \end{vmatrix} = -2a^4$$

$$\begin{vmatrix} 3 & a^2 & 5a^2 \\ 3a & 4 & 6 \\ 4a^2 & 4a & 6a \end{vmatrix} = \begin{vmatrix} 5 & 3a^2 & 5a^2 \\ 5a & 4 & 6 \\ 4a^2 & 4a & 6a \end{vmatrix} = -2a^4$$

$$\begin{vmatrix} 1 & a^2 & 5a^2 \\ a & 2 & 6 \\ 2a^2 & 2a & 6a \end{vmatrix} = \begin{vmatrix} 5 & a^2 & 5a^2 \\ 5a & 2 & 6 \\ 6a^2 & 2a & 6a \end{vmatrix} = -4a^4$$

Thus each pair of these three lines meet. The meeting points for the pairs of lines are, for

$$\text{Line 1 and Line 2; } \begin{pmatrix} 3a^3 - 3 \\ 7a - \frac{8}{a^2} \\ 8a^2 - \frac{8}{a} \end{pmatrix}, \text{Line 2 and Line 3; } \begin{pmatrix} 15a^3 - 15 \\ 23a - \frac{24}{a^2} \\ 24a^2 - \frac{24}{a} \end{pmatrix}$$

$$\text{Line 3 and Line 1; } \begin{pmatrix} 5a^3 - 5 \\ 11a - \frac{12}{a^2} \\ 12a^2 - \frac{12}{a} \end{pmatrix}$$

Level 2

The equation of the plane can be written as

$$\underline{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

where $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is a vector that is on the plane, and $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ are vectors lying in the plane.

So here is our first equation for the plane;

$$\underline{r} = \begin{pmatrix} 1 \\ a \\ 2a^2 \end{pmatrix} + \lambda \begin{pmatrix} a^2 \\ 2 \\ 2a \end{pmatrix} + \mu \begin{pmatrix} 3a^2 \\ 4 \\ 4a \end{pmatrix}.$$

To find its equation in another way, we can write;

$$x = 1 + \lambda a^2 + 3\mu a^2, y = a + 2\lambda + 4\mu, z = 2a^2 + 2a\lambda + 4\mu a.$$

Eliminating λ and μ gives us $-ay + z = a^2$. We can write this as;

$$\underline{r}. \begin{pmatrix} 0 \\ -a \\ 1 \end{pmatrix} = a^2.$$

Level 3

The x-coordinates are only equal if $a = 1$, when the three lines all meet at $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$.

Level 4

As a increases, the a^3 term dominates in the coordinates of the meeting points. This means that the meeting points become increasingly close to being

$$\begin{pmatrix} 3a^3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 15a^3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5a^3 \\ 0 \\ 0 \end{pmatrix},$$

and these are points on a straight line. Thus one of the angles in the triangle tends towards π.

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Further Risp 37: Complex Loci: Notes

Topic: loci in the Argand diagram
Type of task: introductory
Preliminary knowledge required: basic complex numbers
Computing needs: optional (graphing package)

Level 1

The curve $|z - p| = q$ is going to be a circle, centre p and radius q. We can see the equation fixes the distance of z from p to be q.

The curve $\arg(z - p) = q$ is a half-line (or ray) starting at p, and heading off to infinity at an angle of q with the positive horizontal. We can see from the equation that the angle the line segment from p to z must make an angle q with the positive x-direction.

Level 2

A is a circle centre u, radius $|v|$, while B is also a circle, centre v, radius $|u|$.

Level 3

We have that $u = a + bi$, while $v = c + di$.

The curves $|z - u| = |v|$, and $|z - v| = |u|$ are both circles, with centres u and v respectively.

We can be sure that A and B always touch or cross, since $u + v$ is clearly on both circles.

Do A and B ever touch? The circles are (in Cartesian coordinates) $(x - a)^2 + (y - b)^2 = c^2 + d^2$, and $(x - c)^2 + (y - d)^2 = a^2 + b^2$.

They will touch if the distance from u to v is the sum of their radii, that is, if $\sqrt{(a - c)^2 + (b - d)^2} = \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2}$.

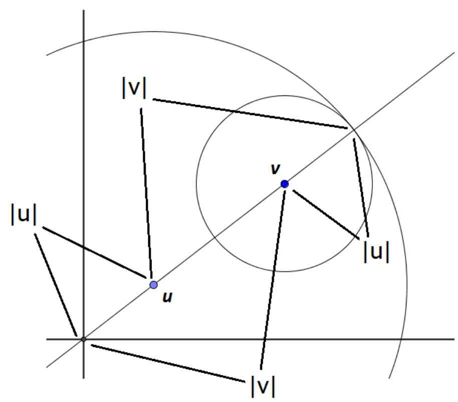
Squaring, we have $(a-c)^2+(b-d)^2 = a^2+b^2+c^2+d^2+2\sqrt{(a^2+b^2)(c^2+d^2)}$, which gives $-(ac+bd) = \sqrt{(a^2+b^2)(c^2+d^2)}$.

Squaring again, we have $a^2c^2 + b^2d^2 + 2abcd = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$.

Cancelling and simplifying gives $(ac-bd)^2 = 0$, and so $ac = bd$, or $\dfrac{b}{a} = \dfrac{d}{c}$.

This means that u, v and the origin must lie in a straight line, or $u = \lambda v$.

A little geometrical reflection tells us that if $u = \lambda v$, then the circles will surely touch (internally).

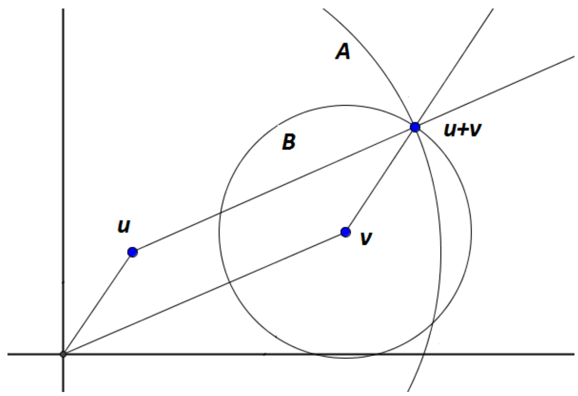


So we now have an iff statement; the circles $|z-u| = |v|$, and $|z-v| = |u|$ touch if and only if $u = \lambda v$.

If one of the circles goes through the origin, then $|u| = |v|$, and so both circles go through the origin. The converse is also true.

Level 4

The curves $\arg(z-u) = \arg(v)$ and $\arg(z-v) = \arg(u)$ are both half-lines, and clearly $u+v$ will be on both. This point is on both A and B as well.



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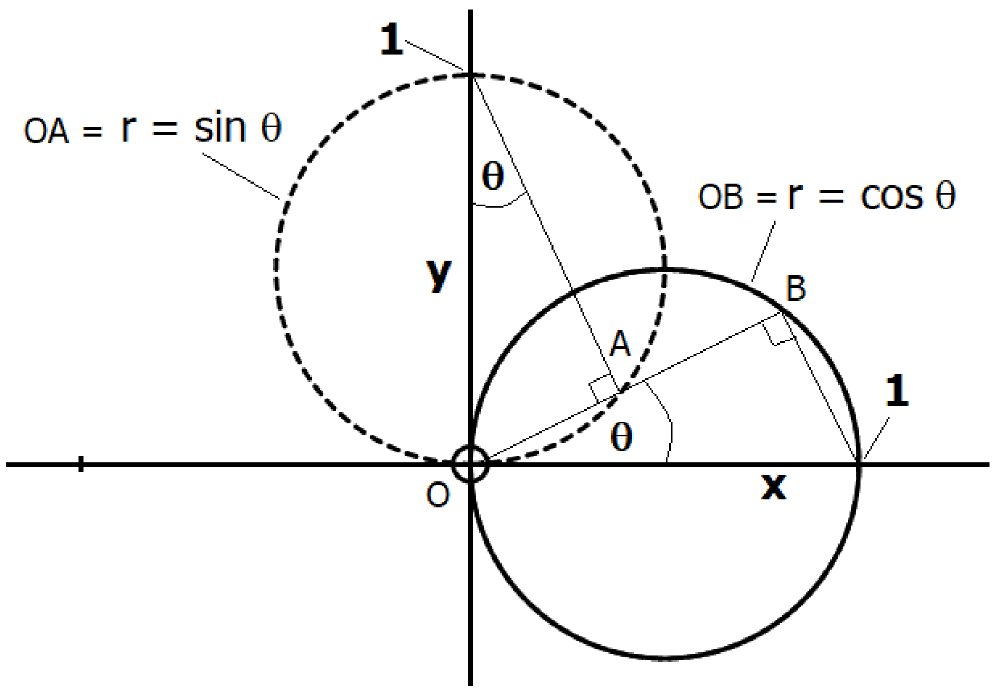
Further Risp 38: A Reduction Formula: Notes

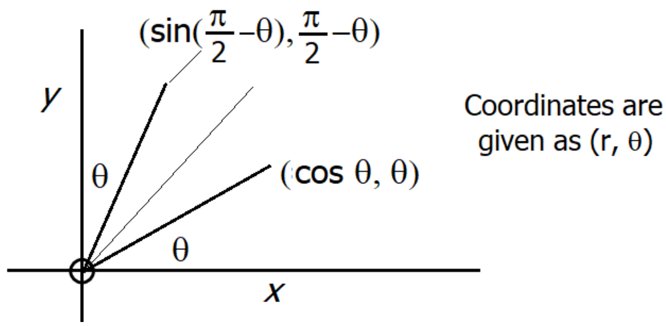
Topic: reduction formulae
Type of task: introductory
Preliminary knowledge required: integration of trig functions, polar area
Computing needs: a graphing package

Level 1

Both $r = \cos\theta$ and $r = \sin\theta$ are circles, both with radius 0.5. A diagram makes it clear why this is so.



How can we find the area of overlap here?



There is certainly symmetry in the line $\theta = \dfrac{\pi}{4}$, since $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$. For every point (r, θ) on $r = \cos\theta$, there is the point $\left(r, \sin\left(\frac{\pi}{2} - \theta\right)\right)$ on $r = \sin\theta$. Thus we have symmetry in $\theta = \dfrac{\pi}{4}$, and to find the area of overlap O, we can find half the area and then double this.

Using the formula for for a polar curve, area $= \dfrac{1}{2}\int_\alpha^\beta r^2 d\theta$, and we have $O = 2 \times \dfrac{1}{2}\int_0^{\pi/4} \cos^2\theta d\theta$. Thus

$$O = \int_0^{\pi/4} \frac{1+\cos(2\theta)}{2} d\theta = \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{4} = 0.1427...$$

Level 2

The graphs of $r = \sin^n\theta$ and $\cos^n\theta$ are not circles for $n > 1$, but they always cross, since $\theta = \dfrac{\pi}{4} \Rightarrow \cos^n\theta = \sin^n\theta = 2^{n/2}$. Alternatively, the tangent to $r = \cos^n\theta$ at the origin is the y-axis, and the tangent to $r = \sin^n\theta$ at the origin is the x-axis, so once again, there must be an overlap.

Let's call the area of overlap in this case I_n. This area appears to decrease as n increases.

Level 3

We can now see that $I_n = 2\times\frac{1}{2}\int_0^{\pi/4}\cos^{2n}\theta d\theta$. Using integration by parts;

$I_n = [\sin\theta\cos^{2n-1}\theta]_0^{\pi/4} - \int_0^{\pi/4}\sin\theta(-\sin\theta)(2n-1)\cos^{2n-2}\theta d\theta$

$= \left(\frac{1}{\sqrt{2}}\right)^{2n} + \int_0^{\pi/4}(1-\cos^2\theta)(2n-1)\cos^{2n-2}\theta d\theta$

$= 2^{-n} + (2n-1)I_{n-2} - (2n-1)I_n.$

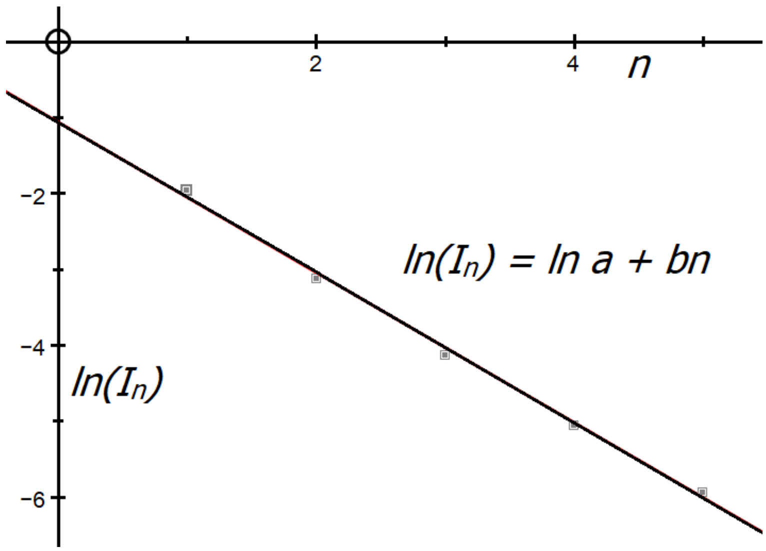
So we have $I_n = \frac{2n-1}{2n}I_{n-2} - \frac{1}{2^{n+1}n}$, and this is our reduction formula.

Level 4

We have already seen that $I_1 = 0.1427...$ Using this, we have

$$\begin{aligned}
I_2 &= \frac{3}{4}(0.1427) - \frac{1}{16} = 0.04453...\\
I_3 &= \frac{5}{6}(0.04453) - \frac{1}{48} = 0.01628...\\
I_4 &= \frac{7}{8}(0.01628) - \frac{1}{128} = 0.006433...\\
I_5 &= \frac{9}{10}(0.006433) - \frac{1}{320} = 0.002664...
\end{aligned}$$

If $I_n \approx ae^{bn}$, then $\ln(I_n) = \ln a + bn$. Plotting $\ln(I_n)$ against n gives roughly a straight line here, with $a \approx \frac{1}{e}, b \approx -1$.



Thus $I_n = e^{-n-1}$ appears to be a reasonable model for I_n for small n.

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Further Risp 39: Friendly Rational Functions: Notes

Topic: integration
Type of task: introductory
Preliminary knowledge required:
division of polynomials, integration techniques
Computing needs: none

Level 1

We have $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$.

Level 2

If $f(x) = x^3 + 3x^2 + 12x + 10$, then $g(x) = \frac{3x^2 + 6x + 12}{x^3 + 3x^2 + 12x + 10}$, and $\int g(x)dx = \ln|x^3 + 3x^2 + 12x + 10| + c$.

We also have $\frac{1}{g(x)} = \frac{x^3 + 3x^2 + 12x + 10}{3x^2 + 6x + 12} = \frac{x}{3} + \frac{1}{3} + \frac{6x + 6}{3x^2 + 6x + 12}$.

Note the final fraction is of the form $\frac{u'(x)}{u(x)}$, so $\frac{1}{g(x)}$ integrates easily to $\frac{x^2}{6} + \frac{x}{3} + \ln|3x^2 + 6x + 12| + c$.

Level 3

Let $f(x) = ax^2 + bx + c$, so we have $g(x) = \frac{2ax + b}{ax^2 + bx + c}$.

Now $\frac{1}{g(x)} = \frac{ax^2 + bx + c}{2ax + b}$, which on dividing out gives us

$$\frac{x}{2} + \frac{b}{4a} + \frac{c - \frac{b^2}{4a}}{2ax + b}.$$

If this final fraction is to be simple, we need $2a = c - \frac{b^2}{4a} \Rightarrow c = \frac{8a^2 + b^2}{4a}$. This is the condition for f to be friendly.

We can also note that if $b = 2a$, then $c = 3a$, and so $f(x) = x^2 + 2x + 3$ is arguably the simplest friendly quadratic.

Level 4

Let $f(x) = x^4 + 4x^3 + cx^2 + dx + e$.

So we need to find $\dfrac{x^4 + 4x^3 + cx^2 + dx + e}{4x^3 + 12x^2 + 2cx + d}$. On dividing out, our final fraction in $\dfrac{1}{g(x)}$ becomes

$$\frac{(\frac{c}{2} - \frac{48}{16})x^2 + (\frac{3d}{4} - \frac{4c}{8})x + (e - \frac{4d}{16})}{4x^3 + 12x^2 + 2cx + d}$$

So we need $12 = \dfrac{c}{2} - \dfrac{48}{16}$, and $24 = \dfrac{3d}{4} - \dfrac{4c}{8}$, and $2c = e - \dfrac{4d}{16}$.

These yield $(c, d, e) = (30, 52, 73)$, and the polynomial $f(x)$ is

$$x^4 + 4x^3 + 30x^2 + 52x + 73.$$

So what differential equation must $f(x)$ satisfy if it is friendly?

We need $\dfrac{1}{g(x)} = px + q + h(x)$, where $h(x)$ is simple.

Dividing out in the general case (where $f(x) = ax^n + bx^{n-1} + ...$ is of degree n) tells us that $p = \dfrac{1}{n}, q = \dfrac{b}{n^2 a}$. So we have

$$\frac{f(x)}{f'(x)} = \frac{x}{n} + \frac{b}{n^2 a} + \frac{f''(x)}{f'(x)}.$$

Putting $y = f(x)$;

$$\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{x}{n} + \frac{b}{n^2 a}\right) - y = 0.$$

Level 5

We are given that $f(x)$ integrates to $g(x)$, and that also $\frac{1}{f(x)}$ integrates to $\frac{1}{g(x)}$. Differentiating, we have

$$f(x) = g'(x) \Rightarrow \frac{1}{f(x)} = -\frac{1}{(g(x))^2}g'(x) = \frac{1}{g'(x)}.$$

Let $y = g(x)$. Thus

$$y^2 = -\left(\frac{dy}{dx}\right)^2 \Rightarrow y = \pm i\frac{dy}{dx} \Rightarrow \int \pm i dx = \int \frac{1}{y}dy,$$

which gives us that $\ln y = \pm ix + c$, and so $Ae^{\pm ix}$ is $g(x)$ for some constant A, which gives either (for some constant B)

$$f(x) = Be^{ix}, g(x) = -iBe^{\pm ix}$$

or

$$f(x) = Be^{-ix}, g(x) = iBe^{\pm ix}.$$

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Further Risp 40: Adding a Rotation to a Reflection: Notes

Topic: geometry
Type of task: introductory
Preliminary knowledge required: basic trigonometry and matrix theory
Computing needs: none

Level 1

The matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ represents a rotation through θ anti-clockwise about the origin. It has determinant 1.

The matrix $B = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ represents a reflection in the line $y = x\tan\frac{\alpha}{2}$. It has a determinant of -1.

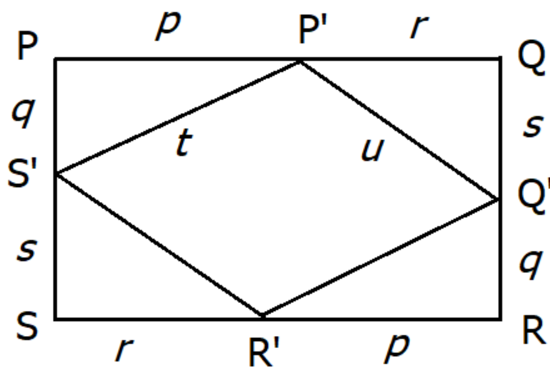
Adding these gives $A + B = \begin{pmatrix} \cos\theta + \cos\alpha & -\sin\theta + \sin\alpha \\ \sin\theta + \sin\alpha & \cos\theta - \cos\alpha \end{pmatrix}$, which becomes $\begin{pmatrix} 2\cos\theta & 0 \\ 2\sin\theta & 0 \end{pmatrix}$ when $\theta = \alpha$. This maps the point $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} 2x\cos\theta \\ 2x\sin\theta \end{pmatrix}$, and the plane is mapped onto the line $y = x\tan\theta$. Every point on the line $x = k$ maps to the the point $(2k\cos\theta, 2k\sin\theta)$.

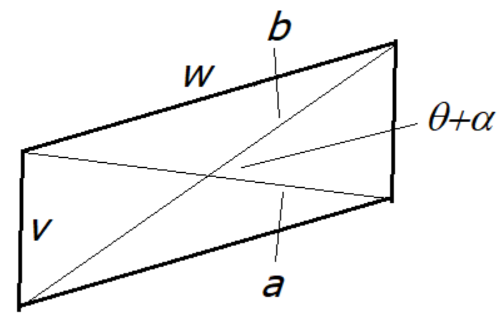
Level 2

We have that the matrix $aA+bB = \begin{pmatrix} a\cos\theta + b\cos\alpha & -a\sin\theta + b\sin\alpha \\ a\sin\theta + b\sin\alpha & a\cos\theta - b\cos\alpha \end{pmatrix}$, and the determinant here is $a^2 - b^2$, which is independent of both θ and α.

Levels 3 and 4

The rectangle and the parallelogram look like this (drawings not to scale);





By Pythagoras in the rectangle,

$$u^2 = r^2 + s^2 = a^2 + b^2 + 2ab(\sin\theta\sin\alpha - \cos\theta\cos\alpha) = a^2 + b^2 - 2\cos(\theta + \alpha),$$

and

$$t^2 = p^2+q^2 = a^2+b^2+2ab(-\sin\theta\sin\alpha+\cos\theta\cos\alpha) = a^2+b^2-2ab\cos(\pi-(\theta+\alpha)).$$

By the cosine rule in the parallelogram,

$$v^2 = a^2 + b^2 - 2ab\cos(\theta + \alpha), w^2 = a^2 + b^2 - 2ab\cos(\pi - (\theta + \alpha)),$$

and so $v = t, w = u$, and the parallelogram will fit (after possible adjustment of angles) to take up the shape $P'Q'R'S'$.

Extra question; is there ever a case where adjusting the angles will not be necessary here?

Level 5

If p, q, r and s are all positive, then $b\sin\alpha > a\sin\theta > 0 \Rightarrow \dfrac{b}{a} > \dfrac{\sin\theta}{\sin\alpha} > 0$.

Also $a\cos\theta > b\cos\alpha > 0 \Rightarrow \dfrac{a}{b} > \dfrac{\cos\alpha}{\cos\theta} > 0$.

Multiplying these inequalities gives

$$1 > \frac{\tan\theta}{\tan\alpha} \Rightarrow \tan\alpha > \tan\theta \Rightarrow \alpha > \theta.$$

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Biography

Jonny Griffiths taught maths at Paston Sixth Form College in Norfolk for over twenty years, being a Gatsby Teacher Fellow for the year 2005-6, when he created the popular *Risps* website. He has studied mathematics and education at Cambridge University, the Open University, and at the University of East Anglia.

Claims to fame include singing with the 1980s band *Harvey and the Wallbangers*, and playing the character of Stringfellow on *Playdays*, the television programme for children. He has had a theological book called *The 100-Word Bible* published by Darton, Longman and Todd.

An ebook of stories from his classroom was published by the Association of Teachers of Mathematics under the title *Correlation Street*, and he has also written many articles for the *Times Educational Supplement*. The ATM has also published his ebook on proof, *The Proving Ground*.

He has worked recently for Underground Mathematics, Hodder, HarperCollins, MEI and Integral. He was the originator and author of the A Level Maths competition *Ritangle* in 2016, which has run every year since.

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